LINEAR KOSZUL DUALITY II – COHERENT SHEAVES ON PERFECT SHEAVES

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Abstract. In this paper we continue the study (initiated in [MR1]) of linear Koszul duality, a geometric version of the standard duality between modules over symmetric and exterior algebras. We construct this duality in a very general setting, and prove its compatibility with morphisms of vector bundles and base change.

Introduction

0.1. In [MR1] we have defined and initiated the study of *linear Koszul duality*, a geometric version of the standard Koszul duality between (graded) modules over the symmetric algebra of a vector space V and (graded) modules over the exterior algebra of the dual vector space V^* (see e.g. [BGG, GKM]). In our setting of [MR1], we replaced the vector space V by a 2-term complex of locally free sheaves on a (smooth) scheme X, and V^* by a shift of the dual complex, and obtained an equivalence of derived categories of dg-modules over the symmetric algebras (in the *graded* sense) of these complexes. As an application, given a vector bundle E over X and subbundles $F_1, F_2 \subset E$, for an appropriate choice of a 2-term complex we obtained an equivalence of derived categories of (dg-)sheaves on the derived intersections $F_1 \overset{R}{\cap}_E F_2$ and $F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp$. Here E^* is the dual vector bundle, and $F_1^\perp, F_2^\perp \subset E^*$ are the orthogonals to F_1 and F_2. A version of this equivalence (when $F_2 = E$) has been used by the second author in [Ri] to construct a Koszul duality for representations of semi-simple Lie algebras in positive characteristic. This construction is also used to study a more geometric question in [UI], see [AG, Appendix F] for comments.

The main result of this paper is a further generalization of this equivalence, where now V is replaced by a finite complex of locally free sheaves of arbitrary length (in non-positive degrees), on a scheme satisfying reasonable assumptions. (Schemes satisfying these conditions are called *nice*, see §1.1.) See Theorems 1.4.1, 1.6.1 and 1.7.1 for three versions of this equivalence. This generalization uses ideas of Positselski ([Po]), and does not rely on [MR1] except for some technical lemmas.

0.2. If $\mathcal{E}$ is a complex of locally free sheaves of $\mathcal{O}_X$-modules, then it is well known that the category of quasi-coherent $\mathrm{S}_{\mathcal{O}_X}(\mathcal{E}^\vee)$-modules is equivalent to the category of quasi-coherent sheaves on the (unique) vector bundle whose sheaf of sections is $\mathcal{E}$. By analogy, if $\mathcal{X}$ is a finite complex of locally free $\mathcal{O}_X$-modules, then the derived category of quasi-coherent dg-modules over the (graded) symmetric algebra $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{X}^\vee)$ can be considered as the derived category of quasi-coherent sheaves on the complex $\mathcal{X}$. So to any bounded complex $\mathcal{X}$ of locally free $\mathcal{O}_X$-modules on a nice scheme X (in non-positive degrees) one can associate a derived category of quasi-coherent sheaves on $\mathcal{X}$, and in this terminology linear Koszul duality is an equivalence between certain categories of quasi-coherent sheaves on $\mathcal{X}$ and on $\mathcal{X}^\vee[-1]$.

Assume moreover that X is a scheme over a field of characteristic zero. Then if $\mathcal{X}$ is a bounded complex of locally free sheaves, the complex $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{X})$ is a direct factor of the tensor algebra $\mathrm{T}_{\mathcal{O}_X}(\mathcal{X})$; it follows that if $\mathcal{X} \to \mathcal{X}'$ is a quasi-isomorphism of such complexes, the induced morphism $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{X}) \to \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{X}')$ is a quasi-isomorphism[1] (since such property is clear for the tensor algebra). If X moreover admits an ample family of line bundles, then using [SGA6, Lemme 2.2.8.c] we deduce that for any perfect sheaf $\mathcal{F}$ in $\mathcal{D}^b\mathsf{Coh}(X)$ one can define the derived category of quasi-coherent sheaves on $\mathcal{F}$ (which is well defined up to equivalence; i.e. this category does not depend on the presentation of $\mathcal{F}$ as a bounded complex of locally free sheaves), and then one can interpret linear Koszul duality in these terms. This justifies our title.

0.3. In Sections 2 and 3 we study the behaviour of this linear Koszul duality under natural operations: we prove compatibility results with respect to morphisms of perfect sheaves and base change. We will use these results in [MR2] to construct a geometric realization of the Iwahori–Matsumoto involution of affine Hecke algebras.

These compatibility properties are inspired by the results of the second author in [Ri, §2], and are quite similar to some compatibility properties of the Fourier transform on constructible sheaves (see [KS, §3.7]). In fact, we will make this similarity precise in [MR3], showing that our linear Koszul duality and a certain Fourier transform isomorphism due to Kashiwara are related via the Chern character from equivariant K-theory to (completed) equivariant Borel–Moore homology. This result will explain the relation between the geometric realization of the Iwahori–Matsumoto involution in [MR2] and the geometric realization of the Iwahori–Matsumoto involution for *graded* affine Hecke algebras constructed in [EM]; see [MR3] for details.

0.4. Notation. If X is a scheme, we denote by $\mathsf{Sh}(X)$ the category of all sheaves of $\mathcal{O}_X$-modules. We denote by $\mathsf{QCoh}(X)$, respectively $\mathsf{Coh}(X)$ the category of quasi-coherent, respectively coherent, sheaves on X.

If X is a scheme and $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, we denote by $\mathcal{F} \boxplus \mathcal{G}$ the $\mathcal{O}_{X^2}$-module $(p_1)^*\mathcal{F} \oplus (p_2)^*\mathcal{G}$ on X^2, where $p_1, p_2 : X \times X \to X$ are the first and second projections. If X is a Noetherian scheme and $Y \subseteq X$ is a closed subscheme, we denote by $\mathsf{Coh}_Y(X)$ the full subcategory of $\mathsf{Coh}(X)$ whose objects are supported set-theoretically on Y. We use similar notation for G-equivariant sheaves (where G is an algebraic group acting on X).

We will frequently work with $\mathbb{Z}^2$-graded sheaves $\mathcal{M}$. The (i, j) component of $\mathcal{M}$ will be denoted $\mathcal{M}^i_j$. Here "i" will be called the cohomological grading, and "j" will be called the internal grading. Ordinary sheaves will be considered as $\mathbb{Z}^2$-graded sheaves concentrated in bidegree $(0, 0)$.

[1] Note that this property does not hold over a field k of characteristic p (as pointed out to one of us by P. Polo): for instance the symmetric algebra of the exact complex $(k \xrightarrow{\mathrm{id}_k} k)$ where the first copy is in even degree n and the second one in degree $n + 1$, is the de Rham complex of $\mathbb{A}^1_k$; in particular its cohomology is not k. This gives a simple example of the principle that the theory of commutative dg-algebras is not suitable outside of fields of characteristic zero, which justifies the need for Lurie's formalism of Derived Algebraic Geometry.

As usual, if $\mathcal{M}$ is a $\mathbb{Z}^2$-graded sheaf of $\mathcal{O}_X$-modules, we denote by $\mathcal{M}^\vee$ the $\mathbb{Z}^2$-graded $\mathcal{O}_X$-module such that

$$(\mathcal{M}^\vee)^i_j \ := \ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^{-i}_{-j}, \mathcal{O}_X).$$

As in [MR1] we will work with $\mathbb{G}_\mathbf{m}$-equivariant sheaves of quasi-coherent $\mathcal{O}_X$-dg-algebras over a scheme X (for the trivial $\mathbb{G}_\mathbf{m}$-action on X).[2] If $\mathcal{A}$ is such a dg-algebra, we denote by $\mathcal{C}(\mathcal{A}-\mathsf{Mod})$ the category of $\mathbb{G}_\mathbf{m}$-equivariant quasi-coherent sheaves of $\mathcal{O}_X$-dg-modules over $\mathcal{A}$ (Beware that for simplicity we do not indicate $\mathbb{G}_\mathbf{m}$ or "gr" in this notation, contrary to our conventions in [MR1, Ri].) We denote by $\mathcal{D}(\mathcal{A}-\mathsf{Mod})$ the associated derived category. On a few occasions we will also consider the category $\widetilde{\mathcal{C}}(\mathcal{A}-\mathsf{Mod})$ of *all* sheaves of $\mathbb{G}_\mathbf{m}$-equivariant $\mathcal{A}$-dg-modules on X (in the sense of [Ri, §1.7]), and the associated derived category $\widetilde{\mathcal{D}}(\mathcal{A}-\mathsf{Mod})$.

If X is a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-modules (considered as a bimodule where the left and right actions coincide), we denote by $\mathrm{S}_{\mathcal{O}_X}(\mathcal{F})$, respectively $\bigwedge_{\mathcal{O}_X}(\mathcal{F})$, the symmetric, respectively exterior, algebra of $\mathcal{F}$, i.e. the quotient of the tensor algebra of $\mathcal{F}$ by the relations $f \otimes g - g \otimes f$, respectively $f \otimes g + g \otimes f$ and $f \otimes f$[3], for f, g local sections of $\mathcal{F}$. If $\mathcal{F}$ is a ($\mathbb{G}_\mathbf{m}$-equivariant) complex of $\mathcal{O}_X$-modules, then these algebras are sheaves of ($\mathbb{G}_\mathbf{m}$-equivariant) dg-algebras in a natural way. If $\mathcal{F}$ is a complex of ($\mathbb{G}_\mathbf{m}$-equivariant) $\mathcal{O}_X$-modules, we denote by $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F})$ the graded-symmetric algebra of $\mathcal{F}$, i.e. the quotient of the tensor algebra of $\mathcal{F}$ by the relations $f \otimes g - (-1)^{|f|\cdot|g|} g \otimes f$ for f, g homogeneous local sections of $\mathcal{F}$, together with $f \otimes f$ for local sections f such that $|f|$ is odd.[4] Again, this algebra is a sheaf of ($\mathbb{G}_\mathbf{m}$-equivariant) dg-algebras in a natural way.

We will use the general convention that we denote similarly a functor between two categories and the induced functor between the opposite categories.

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1. Linear Koszul duality

1.1. Nice schemes. Most of our results will be proved under some technical assumptions on our base scheme. To simplify the statements we introduce the following terminology.

Definition 1.1.1. A scheme is *nice* if it is separated, Noetherian and of finite Krull dimension.

[2] In a different terminology one would replace "$\mathbb{G}_\mathbf{m}$-equivariant dg-algebra" by "dgg-algebra" (where dgg stands for differential graded graded).

[3] Of course, this second set of equations is redundant if 2 is invertible in $\Gamma(X, \mathcal{O}_X)$.

[4] Again, this second set of equations is redundant if 2 is invertible in $\Gamma(X, \mathcal{O}_X)$.

With this terminology, the schemes considered in [BR, §3.1] are exactly the nice schemes. In particular, for every graded-commutative, non-positively graded $\mathcal{O}_X$-dg-algebra $\mathcal{A}$, the category of sheaves of $\mathcal{A}$-dg-modules has enough K-flat and K-injective objects in the sense of [Sp] and, if $\mathcal{A}$ is moreover quasi-coherent, the category of quasi-coherent sheaves of $\mathcal{A}$-dg-modules has enough K-flat and K-injective objects. If $\mathcal{A}$ is $\mathbb{G}_\mathbf{m}$-equivariant, then similar results hold for the categories $\widetilde{\mathcal{C}}(\mathcal{A}-\mathsf{Mod})$ and $\mathcal{C}(\mathcal{A}-\mathsf{Mod})$. (The case of $\widetilde{\mathcal{C}}(\mathcal{A}-\mathsf{Mod})$ is discussed in [Ri, §1.7]; the case of $\mathcal{C}(\mathcal{A}-\mathsf{Mod})$ follows, using the techniques of [BR, §3].) It follows that the usual functors (direct and inverse image, tensor product) admit derived functors. Moreover, these functors admit all the compatibility properties one might expect (see [BR, §3] for precise statements).

Recall finally that under our assumptions the category $\mathcal{D}(\mathcal{A}-\mathsf{Mod})$ depends on $\mathcal{A}$ only up to quasi-isomorphism (see [BR, §3.6]).

1.2. Koszul duality functors on the level of complexes. Let $(X, \mathcal{O}_X)$ be a scheme. We consider a finite complex

$$\mathcal{X} := \ \cdots 0 \to \mathcal{V}^{-n} \to \mathcal{V}^{-n+1} \to \cdots \to \mathcal{V}^0 \to 0 \cdots$$

where $n \geq 0$ and each $\mathcal{V}^i$ is a locally free $\mathcal{O}_X$-module of finite rank ($j \in [\![0, n]\!]$). More precisely, we will consider $\mathcal{X}$ as a complex of graded $\mathcal{O}_X$-modules, where each $\mathcal{V}^i$ is in internal degree 2. Consider also the complex $\mathcal{Y}$ of graded $\mathcal{O}_X$-modules which equals $\mathcal{X}^\vee[-1]$ as a bigraded $\mathcal{O}_X$-module, and where the differential is defined in such a way that

$$d_\mathcal{Y}(y)(v) = (-1)^{|y|} y\big(d_\mathcal{X}(v)\big)$$

for y a local section of $\mathcal{Y}$ and v a local section of $\mathcal{X}$. We define the (graded-commutative) $\mathbb{G}_\mathbf{m}$-equivariant dg-algebras

$$\mathcal{T} := \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{X}), \qquad\qquad \mathcal{S} := \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{Y}).$$

Our goal in this subsection is to construct (covariant) functors

$$\mathscr{A} : \ \mathcal{C}(\mathcal{T}-\mathsf{Mod}) \to \mathcal{C}(\mathcal{S}-\mathsf{Mod}), \qquad \mathscr{B} : \ \mathcal{C}(\mathcal{S}-\mathsf{Mod}) \to \mathcal{C}(\mathcal{T}-\mathsf{Mod}).$$

Let $\mathcal{M}$ be in $\mathcal{C}(\mathcal{T}-\mathsf{Mod})$. We define $\mathscr{A}(\mathcal{M})$ as follows. As a $\mathbb{Z}^2$-graded $\mathcal{O}_X$-module, $\mathscr{A}(\mathcal{M})$ equals $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M}$. The $\mathcal{S}$-action is induced by the left multiplication of $\mathcal{S}$ on itself, and the differential is the sum of two terms, denoted d_1 and d_2. First, d_1 is the natural differential on the tensor product $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M}$, defined by

$$d_1(s \otimes m) = d_\mathcal{S}(s) \otimes m + (-1)^{|s|} s \otimes d_\mathcal{M}(m).$$

Then, consider the natural morphism $i : \mathcal{O}_X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{X}) \cong \mathcal{X}^\vee \otimes \mathcal{X}$. The differential d_2 is the composition of the morphism

$$\left\{ \begin{array}{ccc} \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M} & \to & \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M} \\ s \otimes m & \mapsto & (-1)^{|s|} s \otimes m \end{array} \right.$$

followed by the morphism induced by i

$$\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{X}^\vee \otimes_{\mathcal{O}_X} \mathcal{X} \otimes_{\mathcal{O}_X} \mathcal{M},$$

and finally followed by the morphism

$$\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{X}^\vee \otimes_{\mathcal{O}_X} \mathcal{X} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M}$$

induced by the right multiplication $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{X}^\vee \to \mathcal{S}$ and the morphism $\mathcal{X} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$ induced by the $\mathcal{T}$-action. Locally, one can choose a basis $\{x_\alpha\}$ of $\mathcal{X}$ over $\mathcal{O}_X$, and the dual basis $\{x_\alpha^*\}$ of $\mathcal{X}^\vee$; then d_2 can be described as

$$d_2(s \otimes m) = (-1)^{|s|} \sum_\alpha s x_\alpha^* \otimes x_\alpha \cdot m.$$

The following lemma is proved by a direct computation left to the reader.

Lemma 1.2.1. *These data provide $\mathscr{A}(\mathcal{M})$ the structure of an $\mathcal{S}$-dg-module.*

The functor $\mathscr{B}$ is constructed similarly. For $\mathcal{N}$ in $\mathcal{C}(\mathcal{S}-\mathsf{Mod})$, $\mathscr{B}(\mathcal{N})$ is equal, as a $\mathbb{Z}^2$-graded $\mathcal{O}_X$-module, to $\mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{N}$. The $\mathcal{T}$-module structure is induced by the $\mathcal{T}$-action on $\mathcal{T}^\vee$ defined by $(t \cdot \phi)(t') = (-1)^{|t| \cdot |\phi|} \phi(t \cdot t')$. And the differential is the sum of d_1, which is the differential of the tensor product $\mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{N}$, and d_2, the Koszul differential defined locally by

$$d_2(\phi \otimes n) = (-1)^{|\phi|} \sum_\alpha \phi(x_\alpha \cdot -) \otimes x_\alpha^* \cdot n,$$

The following lemma is proved by a direct computation left to the reader.

Lemma 1.2.2. *These data provide $\mathscr{B}(\mathcal{N})$ the structure of a $\mathcal{T}$-dg-module.*

1.3. Generalized Koszul complexes. We consider the *generalized Koszul complexes*

$$\mathcal{K}^{(1)} := \mathscr{A}(\mathcal{T}^\vee), \qquad \mathcal{K}^{(2)} := \mathscr{B}(\mathcal{S}).$$

There exists a natural morphism of $\mathcal{S}$-dg-modules $\mathcal{K}^{(1)} \to \mathcal{O}_X$ (projection to $\mathcal{O}_X$ in bidegree $(0,0)$), and a natural morphism of $\mathcal{T}$-dg-modules $\mathcal{O}_X \to \mathcal{K}^{(2)}$. The following proposition is similar to [MR1, Lemma 2.6.1].

Proposition 1.3.1. *The natural morphism $\mathcal{K}^{(1)} \to \mathcal{O}_X$, respectively $\mathcal{O}_X \to \mathcal{K}^{(2)}$, is a quasi-isomorphism.*

Proof. The $\mathcal{O}_X$-dg-modules $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ are isomorphic under the (graded) exchange of factors in the tensor product. Hence it is sufficient to treat the case of $\mathcal{K}^{(1)}$. We prove the result by induction on n, the case $n = 0$ being well known (see [MR1, §2.3] and the references therein).

Now assume $n > 0$, and denote by $\mathcal{Z}$ the complex

$$\mathcal{Z} := \ \cdots 0 \to \mathcal{V}_{-n+1} \to \cdots \to \mathcal{V}_0 \to 0 \cdots$$

Denote by $\mathcal{K}_{\mathcal{Z}}^{(1)}$ the Koszul complex for $\mathcal{Z}$; by induction, its cohomology is concentrated in degree 0, and equals $\mathcal{O}_X$. To fix notation, assume n is even. Then there is an isomorphism of graded sheaves

$$\mathcal{K}^{(1)} \ \cong \ \bigoplus_{i,j,k} \bigwedge\nolimits^i (\mathcal{V}_{-n})^\vee \otimes_{\mathcal{O}_X} \big(\mathrm{S}^j(\mathcal{V}_{-n})\big)^\vee \otimes_{\mathcal{O}_X} (\mathcal{K}_{\mathcal{Z}}^{(1)})^k,$$

where the term $\bigwedge^i (\mathcal{V}_{-n})^\vee \otimes_{\mathcal{O}_X} \big(\mathrm{S}^j(\mathcal{V}_{-n})\big)^\vee \otimes_{\mathcal{O}_X} (\mathcal{K}_{\mathcal{Z}}^{(1)})^k$ is in degree $k + nj + (n+1)i$. The differential on $\mathcal{K}^{(1)}$ is the sum of four differentials: d_1 induced by the differential of $\mathcal{K}_{\mathcal{Z}}^{(1)}$, d_2 induced by the Koszul differential on the Koszul complex $\bigwedge(\mathcal{V}_{-n})^\vee \otimes_{\mathcal{O}_X} \big(\mathrm{S}(\mathcal{V}_{-n})\big)^\vee$, d_3 induced by $d_{\mathcal{X}}^{-n} : \mathcal{V}_{-n} \to \mathcal{V}_{-n+1}$ acting in $\mathcal{T}^\vee$, and d_4 induced by $d_{\mathcal{X}}^{-n}$ acting in $\mathcal{S}$. The effect of these differentials on the indices can be described as follows:

$$d_1 : k \mapsto k+1, \quad d_2 : \left\{ \begin{array}{ccc} i & \mapsto & i+1 \\ j & \mapsto & j-1 \end{array} \right., \quad d_3 : \left\{ \begin{array}{ccc} j & \mapsto & j+1 \\ k & \mapsto & k-n+1 \end{array} \right., \quad d_4 : \left\{ \begin{array}{ccc} i & \mapsto & i+1 \\ k & \mapsto & k-n \end{array} \right..$$

Hence $\mathcal{K}^{(1)}$ is the total complex of the double complex whose (p,q)-term is

$$\mathcal{A}^{p,q} := \bigoplus_{\substack{p=i+j, \\ q=k+ni+(n-1)j}} \bigwedge\nolimits^i (\mathcal{V}_{-n})^\vee \otimes_{\mathcal{O}_X} \big(\mathrm{S}^j(\mathcal{V}_{-n})\big)^\vee \otimes_{\mathcal{O}_X} (\mathcal{K}_{\mathcal{Z}}^{(1)})^k,$$

and whose differentials are $d' = d_3 + d_4$, $d'' = d_1 + d_2$. By definition we have $\mathcal{A}^{p,q} = 0$ if $q < 0$. Hence there is a converging spectral sequence

$$E_1^{p,q} = \mathcal{H}^q(\mathcal{A}^{p,*}, d'') \ \Rightarrow \ \mathcal{H}^{p+q}(\mathcal{K}^{(1)})$$

(see [MR1, Proposition 2.2.1(ii)]). Hence it is sufficient to prove that the complex $\mathcal{K}^{(1)}$, endowed with the differential $d_1 + d_2$, has cohomology $\mathcal{O}_X$. However, this complex is a tensor product of two complexes of flat, non-positively graded complexes of $\mathcal{O}_X$-modules with cohomology $\mathcal{O}_X$. Hence the result follows from the Künneth formula. □

1.4. Covariant linear Koszul duality. We denote by $\mathcal{C}(\mathcal{T}-\mathsf{Mod}_-)$ the full subcategory of $\mathcal{C}(\mathcal{T}-\mathsf{Mod})$ whose objects have their internal degree which is bounded above (uniformly in the cohomological degree), and by $\mathcal{D}(\mathcal{T}-\mathsf{Mod}_-)$ the corresponding derived category. We define similarly $\mathcal{C}(\mathcal{S}-\mathsf{Mod}_-)$ and $\mathcal{D}(\mathcal{S}-\mathsf{Mod}_-)$. Note that $\mathcal{T}^\vee$ and $\mathcal{S}$ are both concentrated in non-positive internal degrees. In particular, they are objects of the categories just defined. Note also that the natural functors

$$\mathcal{D}(\mathcal{T}-\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{T}-\mathsf{Mod}) \quad \text{and} \quad \mathcal{D}(\mathcal{S}-\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{S}-\mathsf{Mod})$$

are fully faithful, with essential images the subcategories of dg-modules whose cohomology is bounded above for the internal degree (uniformly in the cohomological degree).

The first (covariant) version of our linear Koszul duality is the following theorem.

Theorem 1.4.1. (1) *The functors*

$$\mathscr{A} : \mathcal{C}(\mathcal{T}-\mathsf{Mod}_-) \to \mathcal{C}(\mathcal{S}-\mathsf{Mod}_-), \qquad \mathscr{B} : \mathcal{C}(\mathcal{S}-\mathsf{Mod}_-) \to \mathcal{C}(\mathcal{T}-\mathsf{Mod}_-)$$

are exact, hence induce functors

$$\overline{\mathscr{A}} : \mathcal{D}(\mathcal{T}-\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{S}-\mathsf{Mod}_-), \qquad \overline{\mathscr{B}} : \mathcal{D}(\mathcal{S}-\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{T}-\mathsf{Mod}_-).$$

(2) *The functors $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$ are equivalences of triangulated categories, quasi-inverse to each other.*

Proof. The idea of the proof is taken from [Po, proof of Theorem A.1.2].

(1) We prove that $\mathscr{A}$ sends acyclic $\mathcal{T}$-dg-modules in $\mathcal{C}(\mathcal{T}-\mathsf{Mod}_-)$ to acyclic $\mathcal{S}$dg-modules; the proof for $\mathscr{B}$ is similar. Let $\mathcal{M}$ be an acyclic object of $\mathcal{C}(\mathcal{T}-\mathsf{Mod}_-)$. To fix notation, assume that $\mathcal{M}_n = 0$ for $n > 0$. Then for any $m \in \mathbb{Z}$ we have

$$\mathscr{A}(\mathcal{M})_m = \bigoplus_{\substack{i \le 0, j \le 0, \\ m=i+j}} \mathcal{S}_i \otimes_{\mathcal{O}_X} \mathcal{M}_j.$$

Hence $\mathscr{A}(\mathcal{M})_m$ can be obtained from the homogeneous internal degree components of $\mathcal{M}$ by tensoring with some homogeneous internal degree components of $\mathcal{S}$ (which are bounded complexes of flat $\mathcal{O}_X$-modules) and taking shifts and cones a finite number of times. More precisely, this process works as follows, e.g. when m is even: start with $\mathcal{S}_m \otimes \mathcal{M}_0$; the Koszul differential defines a morphism of complexes $\mathcal{S}_{m+2} \otimes \mathcal{M}_{-2}[-1] \to \mathcal{S}_m \otimes \mathcal{M}_0$; take its cone $\mathcal{L}$; then the Koszul differential defines a morphism of complexes $\mathcal{S}_{m+4} \otimes \mathcal{M}_{-4}[-1] \to \mathcal{L}$; take its cone; and continue until $\mathcal{S}_0 \otimes \mathcal{M}_m$ is reached.

As $\mathcal{M}_j$ is acyclic for any j, as the tensor product of an acyclic complex with a bounded above complex of flat modules is acyclic, and as the cone of a morphism between acyclic complexes is acyclic, we deduce that $\mathscr{A}(\mathcal{M})_m$ is acyclic for any m, hence that $\mathscr{A}(\mathcal{M})$ is acyclic.

(2) The functors $\mathscr{A}$ and $\mathscr{B}$ are clearly adjoint; in particular we have adjunction morphisms $\mathscr{A} \circ \mathscr{B} \to \mathrm{Id}$ and $\mathrm{Id} \to \mathscr{B} \circ \mathscr{A}$, hence similar morphisms for the induced functors $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$. We show that the morphism $\overline{\mathscr{A}} \circ \overline{\mathscr{B}} \to \mathrm{Id}$ is an isomorphism; the proof for the morphism $\mathrm{Id} \to \overline{\mathscr{B}} \circ \overline{\mathscr{A}}$ is similar. Let $\mathcal{N}$ be an object of $\mathcal{C}(\mathcal{S}-\mathsf{Mod}_-)$. By a construction similar to that of (1), the homogeneous internal degree components of the cone of the morphism $\mathscr{A} \circ \mathscr{B}(\mathcal{N}) \to \mathcal{N}$ can be obtained from the positive homogeneous internal degree components of $\mathcal{K}^{(1)}$ by tensoring with some homogeneous internal degree components of $\mathcal{M}$ and taking shifts and cones a finite number of times. Hence, using Proposition 1.3.1, it suffices to observe that the tensor product of an acyclic, bounded above complex of flat $\mathcal{O}_X$-modules with any complex of $\mathcal{O}_X$-modules is acyclic (see e.g. [Sp, Proposition 5.7]). $\square$

Remark 1.4.2. So far, we have not used the condition that our dg-modules and dg-comodules are quasi-coherent over $\mathcal{O}_X$, or that $(X, \mathcal{O}_X)$ is a scheme. In fact, Theorem 1.4.1 is true for any commutative ringed space $(X, \mathcal{O}_X)$ and the whole categories of $\mathcal{O}_X$-dg-modules over $\mathcal{T}$ and $\mathcal{S}$ (with the prescribed condition on the grading). Our assumption will be used in §1.6 below.

1.5. Reminder on Grothendieck–Serre duality. Let X be a nice scheme. We assume moreover that there exists a dualizing object Ω in $\mathcal{D}^b\mathsf{Coh}(X)$ (see [H1, p. 258]; see [H1, §V.10] for details about this assumption). Let us choose a bounded below complex $\mathcal{I}_\Omega$ of injective quasi-coherent sheaves on X whose image in $\mathcal{D}^b\mathsf{Coh}(X)$ is Ω. Recall that the components of $\mathcal{I}_\Omega$ are also injective in the category $\mathsf{Sh}(X)$. (This follows from [H1, Theorem II.7.18].)

The functor
$$\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{I}_\Omega) : \mathcal{C}\mathsf{Sh}(X) \to \mathcal{C}\mathsf{Sh}(X)^{\mathrm{op}}$$
is exact, hence induces a functor
$$\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{I}_\Omega) : \mathcal{D}\mathsf{Sh}(X) \to \mathcal{D}\mathsf{Sh}(X)^{\mathrm{op}}.$$
It is well known that under our hypotheses the natural functor
$$\mathcal{D}\mathsf{QCoh}(X) \to \mathcal{D}\mathsf{Sh}(X) \tag{1.5.1}$$
is fully faithful (see e.g. [BR, Proposition 3.1.3] and the references therein), as well as the natural functor
$$\mathcal{D}^b\mathsf{Coh}(X) \to \mathcal{D}\mathsf{QCoh}(X). \tag{1.5.2}$$
Hence we can consider the category $\mathcal{D}^b\mathsf{Coh}(X)$ as a subcategory of $\mathcal{D}\mathsf{Sh}(X)$, hence obtain a functor
$$\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{I}_\Omega) : \mathcal{D}^b\mathsf{Coh}(X) \to \mathcal{D}\mathsf{Sh}(X)^{\mathrm{op}}.$$

It is explained in [H1, p. 257] that this functor factors through a functor

$$\mathrm{D}_\Omega : \mathcal{D}^b\mathsf{Coh}(X) \to \mathcal{D}^b\mathsf{Coh}(X)^{\mathrm{op}}.$$

Using again the fully faithfulness of (1.5.1) and (1.5.2), it is easy to construct a morphism $\varepsilon_\Omega : \mathrm{Id} \to \mathrm{D}_\Omega \circ \mathrm{D}_\Omega$ of endofunctors of the category $\mathcal{D}^b\mathsf{Coh}(X)$. The fact that Ω is a dualizing complex means in particular that ε_Ω is an isomorphism of functors (see [H1, Proposition V.2.1]). In particular, D_Ω is an equivalence of categories.

Now we let $\mathcal{A}$ be a $\mathbb{G}_\mathbf{m}$-equivariant, non-positively (cohomologically) graded, graded-commutative sheaf of $\mathcal{O}_X$-dg-algebras on X. We denote by $\mathcal{D}^{\mathrm{bc}}(\mathcal{A}\text{−}\mathsf{Mod})$ the subcategory of $\mathcal{D}(\mathcal{A}\text{−}\mathsf{Mod})$ with objects the dg-modules $\mathcal{M}$ such that for any $j \in \mathbb{Z}$ the object $\mathcal{M}_j$ of $\mathcal{D}\mathsf{QCoh}(X)$ has bounded and coherent cohomology. Our goal now is to explain the construction of an autoequivalence

$$\mathrm{D}_\Omega^\mathcal{A} : \mathcal{D}^{\mathrm{bc}}(\mathcal{A}\text{−}\mathsf{Mod}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{bc}}(\mathcal{A}\text{−}\mathsf{Mod})^{\mathrm{op}}$$

which is compatible with D_Ω in the natural sense. First, there exists a natural functor

$${}^0\widetilde{\mathrm{D}}_\Omega^\mathcal{A} : \widetilde{\mathcal{C}}(\mathcal{A}\text{−}\mathsf{Mod}) \to \widetilde{\mathcal{C}}(\mathcal{A}\text{−}\mathsf{Mod})^{\mathrm{op}}$$

which sends a dg-module $\mathcal{M}$ to the dg-module whose underlying $\mathbb{G}_\mathbf{m}$-equivariant $\mathcal{O}_X$-dg-module is $\mathcal{H}om(\mathcal{M}, \mathcal{I}_\Omega)$, with $\mathcal{A}$-action defined by

$$(a \cdot \phi)(m) = (-1)^{|a|\cdot|\phi|}\phi(a \cdot m)$$

for a a local section of $\mathcal{A}$, ϕ a local section of $\mathcal{H}om(\mathcal{M}, \mathcal{I}_\Omega)$, and m a local section of $\mathcal{M}$. This functor is exact, hence induces a functor

$$\widetilde{\mathrm{D}}_\Omega^\mathcal{A} : \widetilde{\mathcal{D}}(\mathcal{A}\text{−}\mathsf{Mod}) \to \widetilde{\mathcal{D}}(\mathcal{A}\text{−}\mathsf{Mod})^{\mathrm{op}}.$$

Moreover, it is easy to construct a morphism $\widetilde{\varepsilon}_\Omega^\mathcal{A} : \mathrm{Id} \to \widetilde{\mathrm{D}}_\Omega^\mathcal{A} \circ \widetilde{\mathrm{D}}_\Omega^\mathcal{A}$ of endofunctors of $\widetilde{\mathcal{D}}(\mathcal{A}\text{−}\mathsf{Mod})$. Now by [BR, Proposition 3.3.2] the natural functor $\mathcal{D}(\mathcal{A}\text{−}\mathsf{Mod}) \to \widetilde{\mathcal{D}}(\mathcal{A}\text{−}\mathsf{Mod})$ is fully faithful, hence so is also the natural functor $\mathcal{D}^{\mathrm{bc}}(\mathcal{A}\text{−}\mathsf{Mod}) \to \widetilde{\mathcal{D}}(\mathcal{A}\text{−}\mathsf{Mod})$. Moreover, it is easy to prove using the case of $\mathcal{O}_X$ considered above that $\widetilde{\mathrm{D}}_\Omega^\mathcal{A}$ factors through a functor

$$\mathrm{D}_\Omega^\mathcal{A} : \mathcal{D}^{\mathrm{bc}}(\mathcal{A}\text{−}\mathsf{Mod}) \to \mathcal{D}^{\mathrm{bc}}(\mathcal{A}\text{−}\mathsf{Mod})^{\mathrm{op}}.$$

Using fully faithfulness again the morphism $\widetilde{\varepsilon}_\Omega^\mathcal{A}$ induces a morphism $\varepsilon_\Omega^\mathcal{A} : \mathrm{Id} \to \mathrm{D}_\Omega^\mathcal{A} \circ \mathrm{D}_\Omega^\mathcal{A}$ of endofunctors of $\mathcal{D}^{\mathrm{bc}}(\mathcal{A}\text{−}\mathsf{Mod})$. Finally, using again the case of $\mathcal{O}_X$ one can check that $\varepsilon_\Omega^\mathcal{A}$ is an isomorphism of functors, proving in particular that $\mathrm{D}_\Omega^\mathcal{A}$ is an equivalence.

1.6. Contravariant linear Koszul duality. From now on we assume that X is nice. We assume moreover that there exists a dualizing object Ω in $\mathcal{D}^b\mathsf{Coh}(X)$.

Using the same conventions as in §1.5, we denote by $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}\text{−}\mathsf{Mod}_-)$ the subcategory of the category $\mathcal{D}(\mathcal{T}\text{−}\mathsf{Mod}_-)$ whose objects are the dg-modules $\mathcal{M}$ such that, for any $j \in \mathbb{Z}$, $\mathcal{H}^\bullet(\mathcal{M}_j)$ is bounded and coherent. We define similarly the categories $\mathcal{D}^{\mathrm{bc}}(\mathcal{S}\text{−}\mathsf{Mod}_-)$ (replacing $\mathcal{T}$ by $\mathcal{S}$) and $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}\text{−}\mathsf{Mod}_+)$ (replacing "above" by "below").

Clearly, the functors $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$ restrict to equivalences

$$\overline{\mathscr{A}}^{\mathrm{bc}} : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}\text{−}\mathsf{Mod}_-) \xrightarrow{\sim} \mathcal{D}^{\mathrm{bc}}(\mathcal{S}\text{−}\mathsf{Mod}_-), \qquad \overline{\mathscr{B}}^{\mathrm{bc}} : \mathcal{D}^{\mathrm{bc}}(\mathcal{S}\text{−}\mathsf{Mod}_-) \xrightarrow{\sim} \mathcal{D}^{\mathrm{bc}}(\mathcal{T}\text{−}\mathsf{Mod}_-)$$

(see e.g. the proof of Theorem 1.4.1). On the other hand, in §1.5 we have defined an equivalence $\mathrm{D}^{\mathcal{T}}_{\Omega}$, which induces a conctravariant equivalence of categories

$$\mathbf{D}^{\mathcal{T}}_{\Omega} : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_{+}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_{-})^{\mathrm{op}}.$$

Composing these equivalences we obtain the following result, which is the second (contravariant) version of our linear Koszul duality.

Theorem 1.6.1. *Let X be a nice scheme admitting a dualizing complex Ω. Then the composition $\overline{\mathscr{A}}^{\mathrm{bc}} \circ \mathbf{D}^{\mathcal{T}}_{\Omega}$ gives an equivalence of triangulated categories*

$$K_{\Omega} : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_{+}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{bc}}(\mathcal{S}-\mathsf{Mod}_{-})^{\mathrm{op}},$$

which satisfies $K_{\Omega}(\mathcal{M}[n]\langle m\rangle) = K_{\Omega}(\mathcal{M})[-n]\langle -m\rangle$.

Remark 1.6.2.
1. The equivalence K_{Ω} only depends, up to isomorphism, on the dualizing complex $\Omega \in \mathcal{D}^{b}\mathsf{Coh}(X)$, and not on the injective resolution $\mathcal{I}_{\Omega}$. This justifies the notation.
2. One can describe the equivalence K_{Ω} very explicitly. Namely, if $\mathcal{M}$ is an object of $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_{+})$, the image under K_{Ω} of $\mathcal{M}$ is, as an $\mathcal{S}$-dg-module, the image in the derived category of the complex
$$\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{I}_{\Omega}),$$
where the differential is the sum of the usual differential on the tensor product $\mathcal{S} \otimes \mathcal{H}om(\mathcal{M}, \mathcal{I}_{\Omega})$ and of a Koszul-type differential. (Note that this dg-module might not be quasi-coherent.)
3. The covariant Koszul duality $\overline{\mathscr{A}}$ may seem more appealing than the contravariant duality K_{Ω}. However, the category $\mathcal{D}(\mathcal{T}-\mathsf{Mod}_{-})$ is not very interesting since it does not contain the free module $\mathcal{T}$ in general. In particular, the equivalence $\overline{\mathscr{A}}$ will not yield an equivalence for locally finitely generated dg-modules, contrary to K_{Ω} (see Proposition 1.8.2 below). This equivalence will be interesting, however, if $\mathcal{X}$ is concentrated in odd cohomological degrees. In this case, the equivalence obtained is essentially that of [GKM, Theorem 8.4] (see also [Ri, Theorem 2.1.1]).

1.7. Regraded contravariant linear Koszul duality. Consider the $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra

$$\mathcal{R} := \mathrm{Sym}(\mathcal{Y}[2]).$$

There is a "regrading" equivalence of categories

$$\xi : \mathcal{C}(\mathcal{S}-\mathsf{Mod}) \xrightarrow{\sim} \mathcal{C}(\mathcal{R}-\mathsf{Mod}),$$

which sends the $\mathcal{S}$-dg-module $\mathcal{N}$ to the $\mathcal{R}$-dg-module with (i,j)-component $\xi(\mathcal{M})^i_j := \mathcal{M}^{i-j}_j$. (If one forgets the gradings, the dg-algebras $\mathcal{R}$ and $\mathcal{S}$ coincide, as well as $\mathcal{M}$ and $\xi(\mathcal{M})$. Then the $\mathcal{R}$-action and the differential on $\xi(\mathcal{M})$ are the same as the $\mathcal{S}$-action and the differential on $\mathcal{M}$.) Composing the equivalence of Theorem 1.6.1 with ξ we obtain the third version of our linear Koszul duality, which is the one we will use.

Theorem 1.7.1. *Let X be a nice scheme admitting a dualizing complex Ω. Then the composition $\xi \circ K_{\Omega}$ gives an equivalence of triangulated categories*

$$\kappa_{\Omega} : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_{+}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{bc}}(\mathcal{R}-\mathsf{Mod}_{-})^{\mathrm{op}},$$

which satisfies $\kappa_\Omega(\mathcal{M}[n]\langle m\rangle) = \kappa_\Omega(\mathcal{M})[-n+m]\langle -m\rangle$.

1.8. Finiteness conditions. From now on, for simplicity we assume that $n \leq 1$.

We consider subcategories of locally finitely generated $\mathcal{T}$- and $\mathcal{R}$-dg-modules. More precisely, we let $\mathcal{CFG}(\mathcal{T})$ be the category of locally finitely generated $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{T}$-dg-modules, and $\mathcal{DFG}(\mathcal{T})$ be the associated derived category. We define also $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod})$, respectively $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod}_+)$, as the subcategory of $\mathcal{D}(\mathcal{T}-\mathsf{Mod})$, respectively $\mathcal{D}(\mathcal{T}-\mathsf{Mod}_+)$, whose objects are the dg-modules whose cohomology is locally finitely generated over the cohomology $\mathcal{H}^\bullet(\mathcal{T})$. We use similar notation for $\mathcal{R}$-dg-modules.

The following lemma can be proved as in [MR1, Lemma 3.6.1].

Lemma 1.8.1. *Assume that $n \leq 1$, and that X is a Noetherian scheme.*

(1) *The inclusions $\mathcal{CFG}(\mathcal{T}) \hookrightarrow \mathcal{C}(\mathcal{T}-\mathsf{Mod}_+) \hookrightarrow \mathcal{C}(\mathcal{T}-\mathsf{Mod})$ induce equivalences of triangulated categories*
$$\mathcal{DFG}(\mathcal{T}) \;\cong\; \mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod}_+) \;\cong\; \mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod}).$$
(2) *The inclusions $\mathcal{CFG}(\mathcal{R}) \hookrightarrow \mathcal{C}(\mathcal{R}-\mathsf{Mod}_-) \hookrightarrow \mathcal{C}(\mathcal{R}-\mathsf{Mod})$ induce equivalences of triangulated categories*
$$\mathcal{DFG}(\mathcal{R}) \;\cong\; \mathcal{D}^{\mathrm{fg}}(\mathcal{R}-\mathsf{Mod}_-) \;\cong\; \mathcal{D}^{\mathrm{fg}}(\mathcal{R}-\mathsf{Mod}).$$

Using this lemma, we prove that the equivalence κ_Ω restricts to the subcategories of objects with locally finitely generated cohomology.

Proposition 1.8.2. *Assume X is a nice scheme admitting a dualizing complex Ω, and that $n \leq 1$. Then κ_Ω restricts to an equivalence of triangulated categories*
$$\mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod}_+) \;\cong\; \mathcal{D}^{\mathrm{fg}}(\mathcal{R}-\mathsf{Mod}_-)^{\mathrm{op}}.$$

Proof. First case: $n = 0$. In this case, $\mathcal{T} = \mathrm{S}(\mathcal{V}_0)$, with generators in bidegree $(0,2)$, and $\mathcal{R} = \bigwedge(\mathcal{V}_0)^\vee$, with generators in bidegree $(-1,-2)$. Any object of the category $\mathcal{CFG}(\mathcal{R})$ has a filtration (as an $\mathcal{R}$-dg-module) such that $\mathcal{R}$ acts trivially on the associated graded. Hence, using Lemma 1.8.1, $\mathcal{D}^{\mathrm{fg}}(\mathcal{R}-\mathsf{Mod}_-)$ is generated, as a triangulated category, by trivial $\mathcal{R}$-dg-modules, i.e. by images of objects of $\mathcal{D}^b\mathsf{Coh}(X)$.

On the other hand, we claim that $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod}_+)$ is generated, as a triangulated category, by objects of the form $\mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{F}$, for $\mathcal{F}$ in $\mathcal{D}^b\mathsf{Coh}(X)$ (where the $\mathcal{T}$-action is induced by left multiplication on the left factor). This follows from the following general fact (see [CG, p. 266, last paragraph]), using the classical fact that $\mathcal{DFG}(\mathcal{T})$ is equivalent to the bounded derived category of $\mathbb{G}_{\mathbf{m}}$-equivariant coherent sheaves on the vector bundle over X whose sheaf of sections is $(\mathcal{V}_0)^\vee$.

Proposition 1.8.3. *Let H be an algebraic group, and let $\pi : F \to Y$ be an H-equivariant vector bundle. Then the category $\mathcal{D}^b\mathsf{Coh}^H(F)$ is generated, as a triangulated category, by objects of the form $\pi^*\mathcal{F}$ for $\mathcal{F}$ in $\mathsf{Coh}^H(Y)$.*

We have determined generating subcategory $\mathscr{G}_1$, respectively $\mathscr{G}_2$, for the triangulated category $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod}_+)$, respectively $\mathcal{D}^{\mathrm{fg}}(\mathcal{R}-\mathsf{Mod}_-)$. By definition, κ_Ω induces an equivalence bewteen $\mathscr{G}_1$ and $\mathscr{G}_2$. Hence it induces an equivalence between $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod}_+)$ and $\mathcal{D}^{\mathrm{fg}}(\mathcal{R}-\mathsf{Mod}_-)$.

Second case: $n = 1$. In this case $\mathcal{T} = \mathrm{Sym}(\mathcal{V}_{-1} \to \mathcal{V}_0)$, with $\mathcal{V}_{-1}$ in bidegree $(-1, 2)$ and $\mathcal{V}_0$ in bidegree $(0, 2)$, while $\mathcal{R} = \mathrm{Sym}\big((\mathcal{V}_0)^\vee \to (\mathcal{V}_{-1})^\vee\big)$, with $(\mathcal{V}_0)^\vee$ in bidegree $(-1, -2)$ and $(\mathcal{V}_{-1})^\vee$ in bidegree $(0, -2)$. As in §1.5, we denote by $\mathcal{I}_\Omega$ a bounded below complex of quasi-coherent injective $\mathcal{O}_X$-modules whose image in the derived category is Ω. Let $\mathcal{M}$ be an object of $\mathcal{CFG}(\mathcal{T})$. We have to show that $\kappa_\Omega(\mathcal{M})$ has finitely generated cohomology over $\mathcal{H}^\bullet(\mathcal{R})$ or, equivalently, that its cohomology is locally finitely generated over $\mathrm{S}\big((\mathcal{V}_{-1})^\vee\big)$. In fact it will be equivalent and easier to work with the equivalence K_Ω of Theorem 1.6.1 and the dg-algebra $\mathcal{S}$ (whose generators are in bidegrees $(1, -2)$ and $(2, -2)$).

Let $\widetilde{K}_\Omega$ be the equivalence corresponding to the complex of $\mathcal{O}_X$-modules $\widetilde{\mathcal{X}}$ concentrated in degree 0, with only non-zero component $\mathcal{V}_0$. Denote by $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{S}}$ the dg-algebras defined similarly to $\mathcal{T}$ and $\mathcal{S}$, but for the complex $\widetilde{\mathcal{X}}$ instead of $\mathcal{X}$. With this notation, $K_\Omega(\mathcal{M})$ is the image in the derived category of the $\mathcal{S}$-dg-module $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{I}_\Omega)$ (with a certain differential), and $\widetilde{K}_\Omega(\mathcal{M})$ is the image in the derived category of the $\widetilde{\mathcal{S}}$-dg-module $\widetilde{\mathcal{S}} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{I}_\Omega)$.

The $\mathcal{T}$-dg-module $\mathcal{M}$ is also locally finitely generated as a $\widetilde{\mathcal{T}}$-dg-module. By the first case, we deduce that $\widetilde{K}_\Omega(\mathcal{M})$ has locally finitely generated cohomology over $\widetilde{\mathcal{S}}$; in other words, this cohomology is bounded and coherent over $\mathcal{O}_X$. Now we have an isomorphism of $\mathrm{S}\big((\mathcal{V}_{-1})^\vee\big)$-dg-modules

$$K_\Omega(\mathcal{M}) \;\cong\; \bigoplus_{i,j} \mathrm{S}^i\big((\mathcal{V}_{-1})^\vee\big) \otimes_{\mathcal{O}_X} \big(\widetilde{K}_\Omega(\mathcal{M})\big)^j, \tag{1.8.4}$$

where the term $\mathrm{S}^i\big((\mathcal{V}_{-1})^\vee\big) \otimes_{\mathcal{O}_X} \big(\widetilde{K}_\Omega(\mathcal{M})\big)^j$ is in cohomological degree $j + 2i$. The differential on $K_\Omega(\mathcal{M})$ is the sum of three terms: the differential d_1 induced by $d_{\mathcal{S}}$, the Koszul differential d_2 induced by that of the Koszul complex $(\bigwedge \mathcal{V}_{-1})^\vee \otimes \mathrm{S}\big((\mathcal{V}_{-1})^\vee\big)$, and finally the differential d_3 induced by that of $\widetilde{K}_\Omega(\mathcal{M})$. The effect of these differentials on the degrees of the decomposition (1.8.4) are the following:

$$d_1 : \left\{ \begin{array}{ccc} i & \mapsto & i+1 \\ j & \mapsto & j-1 \end{array} \right., \quad d_2 : \left\{ \begin{array}{ccc} i & \mapsto & i+1 \\ j & \mapsto & j-1 \end{array} \right., \quad d_3 : j \mapsto j+1.$$

Hence $K_\Omega(\mathcal{M})$ is the total complex of the double complex with (p, q)-term

$$\mathcal{B}^{p,q} := \bigoplus_{\substack{p=i,\\ q=j+i}} \mathrm{S}^i\big((\mathcal{V}_{-1})^\vee\big) \otimes_{\mathcal{O}_X} \big(\widetilde{K}_\Omega(\mathcal{M})\big)^j$$

and differentials $d' := d_1 + d_2$, $d'' := d_3$. Now $\mathcal{M}$ is locally finitely generated over $\mathcal{T}$; in particular it is bounded above for the cohomological grading. Hence $\widetilde{K}_\Omega(\mathcal{M})$ is bounded below for the cohomological grading. It follows that $\mathcal{B}^{p,q} = 0$ for $q \ll 0$. Hence by [MR1, Proposition 2.2.1], there exists a converging spectral sequence

$$E_1^{p,q} = \mathcal{H}^q(\mathcal{B}^{p,*}, d'') \;\Rightarrow\; \mathcal{H}^{p+q}(K_\Omega(\mathcal{M})).$$

As $\widetilde{K}_\Omega(\mathcal{M})$ has bounded, coherent cohomology over $\mathcal{O}_X$, E_1 is a locally finitely generated $\mathrm{S}((\mathcal{V}_{-1})^\vee)$-module. Moreover, in the (p, q)-plane, it is concentrated on an ascending diagonal strip. It follows that the spectral sequence is stationary after finitely many steps. We deduce that $\mathcal{H}^\bullet(K_\Omega(\mathcal{M}))$ is locally finitely generated over $\mathrm{S}((\mathcal{V}_{-1})^\vee)$.

By symmetry the equivalence $(\kappa_\Omega)^{-1}$ also sends dg-modules with locally finitely generated cohomology to dg-modules with the same property, which finishes the proof. □

1.9. Intersection of vector subbundles. We can now extend the main result of [MR1] to nice schemes admitting a dualizing complex. We let X be such a scheme, and denote by Ω a dualizing complex.

Let E be a vector bundle over X, let $F_1, F_2 \subset E$ be two vector subbundles, and let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{E}$ be the sheaves of sections of F_1, F_2, E. Let E^* be the dual vector bundle, and $F_1^{\perp}, F_2^{\perp} \subset E^*$ be the orthogonals to F_1 and F_2. We will apply the constructions above to the complex

$$\mathcal{X} := (0 \to \mathcal{F}_1^{\perp} \to \mathcal{F}_2^{\vee} \to 0) \tag{1.9.1}$$

where $\mathcal{F}_1^{\perp}$ is in degree -1, $\mathcal{F}_2^{\vee}$ is in degree 0, and the differential is the composition $\mathcal{F}_1^{\perp} \hookrightarrow \mathcal{E}^{\vee} \twoheadrightarrow \mathcal{F}_2^{\vee}$. We set

$$\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \ := \ \mathcal{D}^{\mathrm{fg}}(\mathcal{T}{-}\mathsf{Mod}), \qquad \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp}) \ := \ \mathcal{D}^{\mathrm{fg}}(\mathcal{R}{-}\mathsf{Mod}).$$

To justify this notation, recall the notion of dg-scheme, first defined in [CK] and later studied in [Ri, MR1, BR]. (Here we follow the conventions of [MR1].) It is explained in [MR1, Lemma 4.1.1] that the derived category of coherent dg-sheaves on the dg-scheme $F_1 \overset{R}{\cap}_E F_2$, respectively $F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp}$, is equivalent to the derived category of quasi-coherent dg-modules over the sheaf of $\mathcal{O}_X$-dg-algebras $\mathcal{T}$, respectively $\mathcal{R}$. (Note that here we consider *usual* dg-modules, and not $\mathbb{G}_{\mathbf{m}}$-equivariant ones.) Hence the category $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}{-}\mathsf{Mod})$, respectively $\mathcal{D}^{\mathrm{fg}}(\mathcal{R}{-}\mathsf{Mod})$, is a "graded version" of this category.

Remark 1.9.2. Our definition of the category $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2)$ is not symmetric in F_1 and F_2. However, it follows from an obvious $\mathbb{G}_{\mathbf{m}}$-equivariant analogue of [MR1, Proposition 1.3.2] that the triangulated categories $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2)$ and $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_2 \overset{R}{\cap}_E F_1)$ are equivalent.

With this notation, and using Lemma 1.8.1, Proposition 1.8.2 can rephrased in the following terms.

Theorem 1.9.3. *Assume that X is a nice scheme admitting a dualizing complex Ω. Then κ_Ω induces an equivalence of triangulated categories denoted similarly*

$$\kappa_\Omega : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \ \xrightarrow{\sim} \ \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp})^{\mathrm{op}},$$

which satisfies $\kappa_\Omega(\mathcal{M}[n]\langle m \rangle) = \kappa_\Omega(\mathcal{M})[-n+m]\langle -m \rangle$.

Remark 1.9.4. One can easily check that, if the assumptions of [MR1, Theorem 4.2.1] are satisfied, then $\mathcal{O}_X$ is a dualizing complex, and the equivalence of *loc. cit.* is isomorphic to the equivalence $\kappa_{\mathcal{O}_X}$ of Theorem 1.9.3.

2. Linear Koszul duality and morphisms of perfect complexes

2.1. Statement. Let us come back to the setting of §1.7. More precisely, we consider a nice scheme X admitting a dualizing complex Ω, and two complexes $\mathcal{X}$ and $\mathcal{X}'$ of locally free sheaves as in §1.2. We denote by $\mathcal{R}, \mathcal{S}, \mathcal{T}$, respectively $\mathcal{R}', \mathcal{S}', \mathcal{T}'$, the dg-algebras constructed from $\mathcal{X}$, respectively $\mathcal{X}'$. We also denote by

$$\kappa_\Omega : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}{-}\mathsf{Mod}_+) \ \xrightarrow{\sim} \ \mathcal{D}^{\mathrm{bc}}(\mathcal{R}{-}\mathsf{Mod}_-)^{\mathrm{op}}, \quad \kappa'_\Omega : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}'{-}\mathsf{Mod}_+) \ \xrightarrow{\sim} \ \mathcal{D}^{\mathrm{bc}}(\mathcal{R}'{-}\mathsf{Mod}_-)^{\mathrm{op}}$$

the associated equivalences of Theorem 1.7.1.

Let $\phi : \mathcal{X}' \to \mathcal{X}$ be a morphism of complexes. This morphism induces morphisms of $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebras

$$\Phi : \mathcal{T}' \to \mathcal{T}, \qquad \Psi : \mathcal{R} \to \mathcal{R}'.$$

In turn, these morphisms of dg-algebras induce functors

$$\Phi_* : \mathcal{C}(\mathcal{T}\text{–}\mathsf{Mod}) \to \mathcal{C}(\mathcal{T}'\text{–}\mathsf{Mod}), \quad \Psi_* : \mathcal{C}(\mathcal{R}'\text{–}\mathsf{Mod}) \to \mathcal{C}(\mathcal{R}\text{–}\mathsf{Mod})$$

(restriction of scalars) and

$$\Phi^* : \mathcal{C}(\mathcal{T}'\text{–}\mathsf{Mod}) \to \mathcal{C}(\mathcal{T}\text{–}\mathsf{Mod}), \quad \Psi^* : \mathcal{C}(\mathcal{R}\text{–}\mathsf{Mod}) \to \mathcal{C}(\mathcal{R}'\text{–}\mathsf{Mod})$$

(extension of scalars).

The functors Φ_* and Ψ_* are exact, hence induce functors

$$R\Phi_* : \mathcal{D}(\mathcal{T}\text{–}\mathsf{Mod}) \to \mathcal{D}(\mathcal{T}'\text{–}\mathsf{Mod}), \quad R\Psi_* : \mathcal{D}(\mathcal{R}'\text{–}\mathsf{Mod}) \to \mathcal{D}(\mathcal{R}\text{–}\mathsf{Mod}).$$

These functors clearly send the subcategory $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}\text{–}\mathsf{Mod}_{\pm})$ into $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}'\text{–}\mathsf{Mod}_{\pm})$ and the subcategory $\mathcal{D}^{\mathrm{bc}}(\mathcal{R}'\text{–}\mathsf{Mod}_{-})$ into $\mathcal{D}^{\mathrm{bc}}(\mathcal{R}\text{–}\mathsf{Mod}_{-})$ (and similarly without "bc").

The functors Φ^* and Ψ^* are not exact. However, it follows from [MR1, Proposition 1.2.3] (existence of K-flat resolutions) that they admit left derived functors

$$L\Phi^* : \mathcal{D}(\mathcal{T}'\text{–}\mathsf{Mod}) \to \mathcal{D}(\mathcal{T}\text{–}\mathsf{Mod}), \quad L\Psi^* : \mathcal{D}(\mathcal{R}\text{–}\mathsf{Mod}) \to \mathcal{D}(\mathcal{R}'\text{–}\mathsf{Mod}).$$

The following result expresses the compatibility of our Koszul duality equivalence κ_Ω with morphisms of perfect sheaves. It is similar in spirit (but in a much more general setting) to [Ri, Proposition 2.5.4].

Proposition 2.1.1. *Let X be a nice scheme admitting a dualizing complex Ω.*

(1) *The functor $L\Psi^*$ resticts to a functor from $\mathcal{D}^{\mathrm{bc}}(\mathcal{R}\text{–}\mathsf{Mod}_{-})$ to $\mathcal{D}^{\mathrm{bc}}(\mathcal{R}'\text{–}\mathsf{Mod}_{-})$, denoted similarly. Moreover, there exists an isomorphism of functors:*

$$L\Psi^* \circ \kappa_\Omega \ \cong \ \kappa'_\Omega \circ R\Phi_* \quad : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}\text{–}\mathsf{Mod}_{+}) \to \mathcal{D}^{\mathrm{bc}}(\mathcal{R}'\text{–}\mathsf{Mod}_{-})^{\mathrm{op}}.$$

(2) *The functor $L\Phi^*$ resticts to a functor from $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}'\text{–}\mathsf{Mod}_{+})$ to $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}\text{–}\mathsf{Mod}_{+})$, denoted similarly. Moreover, there exists an isomorphism of functors:*

$$\kappa_\Omega \circ L\Phi^* \ \cong \ R\Psi_* \circ \kappa'_\Omega \quad : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}'\text{–}\mathsf{Mod}_{+}) \to \mathcal{D}^{\mathrm{bc}}(\mathcal{R}\text{–}\mathsf{Mod}_{-})^{\mathrm{op}}.$$

2.2. Proof of Proposition 2.1.1. To prove Proposition 2.1.1 we need some preparatory lemmas. We assume the conditions in the proposition are satisfied.

Lemma 2.2.1. *The $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module $\mathcal{S}$ is K-flat.*

Proof. It is enough to prove that for any $i \in \mathbb{Z}$, the $\mathcal{O}_X$-dg-module $\mathcal{S}_i$ is K-flat. However, this complex is a bounded complex of flat $\mathcal{O}_X$-modules, which proves this fact. □

Lemma 2.2.2. *For every object $\mathcal{M}$ of $\mathcal{C}(\mathcal{T}\text{–}\mathsf{Mod}_{-})$, there exists an object $\mathcal{M}'$ of $\mathcal{C}(\mathcal{T}\text{–}\mathsf{Mod}_{-})$ which is K-flat as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module and such that the images of $\mathcal{M}$ and $\mathcal{M}'$ in $\mathcal{D}(\mathcal{T}\text{–}\mathsf{Mod}_{-})$ are isomorphic.*

Proof. Let $\mathcal{M}$ be an object of $\mathcal{C}(\mathcal{T}-\mathsf{Mod}_-)$. By Theorem 1.4.1, there exists an object $\mathcal{N}$ of $\mathcal{C}(\mathcal{S}-\mathsf{Mod}_-)$ such that $\mathcal{M}$ and $\mathscr{B}(\mathcal{N})$ are isomorphic in the derived category. Hence we can assume that $\mathcal{M} = \mathscr{B}(\mathcal{N})$.

By the same arguments as in [MR1, Proposition 1.2.3], using the existence of enough flat quasi-coherent $\mathcal{O}_X$-modules in $\mathsf{QCoh}(X)$ (see [BR, §3.2]) one can check that for any object $\mathcal{N}'$ of $\mathcal{C}(\mathcal{S}-\mathsf{Mod}_-)$ there exists an object $\mathcal{N}''$ of $\mathcal{C}(\mathcal{S}-\mathsf{Mod}_-)$ which is K-flat as a $\mathbb{G}_\mathbf{m}$-equivariant $\mathcal{S}$-dg-module and a quasi-isomorphism $\mathcal{N}'' \xrightarrow{\mathrm{qis}} \mathcal{N}'$. Hence we can assume that $\mathcal{N}$ is K-flat as a $\mathbb{G}_\mathbf{m}$-equivariant $\mathcal{S}$-dg-module. Then it follows from Lemma 2.2.1 that $\mathcal{N}$ is also K-flat as a $\mathbb{G}_\mathbf{m}$-equivariant $\mathcal{O}_X$-dg-module (see [Ri, Lemma 1.3.2]).

We claim that, in this case, $\mathscr{B}(\mathcal{N})$ is K-flat as a $\mathbb{G}_\mathbf{m}$-equivariant $\mathcal{O}_X$-dg-module. Indeed, it is enough to show that for any $n \in \mathbb{Z}$, $\mathscr{B}(\mathcal{N})_n$ is a K-flat $\mathcal{O}_X$-dg-module. However, we have

$$\mathscr{B}(\mathcal{N})_n = \bigoplus_{k+l=n} (\mathcal{T}^\vee)_k \otimes \mathcal{N}_l.$$

This sum is finite and, as in the proof of Theorem 1.4.1, this dg-module can be obtained from the dg-modules $(\mathcal{T}^\vee)_k \otimes \mathcal{N}_l$ (usual tensor product) by taking shifts and cones a finite number of times. The latter dg-modules are K-flat over $\mathcal{O}_X$, and the cone of a morphism between K-flat dg-modules is still K-flat, hence these remarks finish the proof of our claim, and also of the lemma. □

Lemma 2.2.3. *If $\mathcal{M}$ is an object of $\mathcal{C}(\mathcal{T}-\mathsf{Mod}_-)$ which is K-flat as a $\mathbb{G}_\mathbf{m}$-equivariant $\mathcal{O}_X$-dg-module, then $\mathscr{A}(\mathcal{M})$ is K-flat as a $\mathbb{G}_\mathbf{m}$-equivariant $\mathcal{S}$-dg-module.*

Proof. We have to check that for any acyclic object $\mathcal{N}$ of $\mathcal{C}(\mathcal{S}-\mathsf{Mod})$, the complex $\mathcal{N} \otimes_\mathcal{S} \mathscr{A}(\mathcal{M})$ is acyclic. Every object of $\mathcal{C}(\mathcal{S}-\mathsf{Mod})$ is a direct limit of objects of $\mathcal{C}(\mathcal{S}-\mathsf{Mod}_-)$ (because $\mathcal{S}$ is concentrated in non-positive internal degrees); moreover if the initial object is acyclic one can choose these objects also acyclic. Hence we can assume that $\mathcal{N}$ is in $\mathcal{C}(\mathcal{S}-\mathsf{Mod}_-)$. Then we have $\mathcal{N} \otimes_\mathcal{S} \mathscr{A}(\mathcal{M}) = \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}$, where the differential is the sum of the usual differential of the tensor product $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}$ and a Koszul-type differential. The same argument as in the proof of Theorem 1.4.1 or Lemma 2.2.2 proves that this complex is acyclic. □

Proof of Proposition 2.1.1. We only prove (1); the proof of (2) is similar.

The regrading functors of §1.7 do not play any role here, hence we will rather work with the equivalences K_Ω, K'_Ω of Theorem 1.6.1. For simplicity, we still denote by $L\Psi^*$ the derived extension of scalars from $\mathcal{S}$- to $\mathcal{S}'$-dg-modules. We denote by $\mathscr{A}'$, $\mathscr{B}'$ the functors of §1.2 relative to the complex $\mathcal{X}'$. It is clear from definition that we have an isomorphism of functors

$$\mathbf{D}_\Omega^{\mathcal{T}'} \circ R\Phi_* \ \cong \ R\Phi_* \circ \mathbf{D}_\Omega^{\mathcal{T}}. \tag{2.2.4}$$

(Here, the first $R\Phi_*$ is considered as a functor from $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_+)$ to $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}'-\mathsf{Mod}_+)$, while the second one is considered as a functor from $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_-)$ to $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}'-\mathsf{Mod}_-)$.)

We claim that there exists an isomorphism of functors from $\mathcal{D}(\mathcal{T}-\mathsf{Mod}_-)$ to $\mathcal{D}(\mathcal{S}'-\mathsf{Mod})$

$$\overline{\mathscr{A}'} \circ R\Phi_* \ \cong \ L\Psi^* \circ \overline{\mathscr{A}}. \tag{2.2.5}$$

First, consider the assigment from $\mathcal{C}(\mathcal{T}-\mathsf{Mod}_-)$ to $\mathcal{C}(\mathcal{S}'-\mathsf{Mod})$ given by $\mathcal{M} \mapsto \Psi^*\mathscr{A}(\mathcal{M}) \cong \mathcal{S}' \otimes_{\mathcal{O}_X} \mathcal{M} \cong \mathscr{A}'\Phi_*\mathcal{M}$. (Here the differential on $\mathcal{S}' \otimes_{\mathcal{O}_X} \mathcal{M}$ involves a Koszul-type differential as usual.) This functor is exact, as the composition of the exact functors Φ_* and $\mathscr{A}'$. The induced functor from $\mathcal{D}(\mathcal{T}-\mathsf{Mod}_-)$ to $\mathcal{D}(\mathcal{S}'-\mathsf{Mod})$ is clearly isomorphic to the left hand side of (2.2.5).

By usual properties of composition of derived functors, we obtain a morphism of functors

$$L\Psi^* \circ \overline{\mathscr{A}} \to \overline{\mathscr{A}'} \circ R\Phi_*. \tag{2.2.6}$$

What we have to show is that this morphism is an isomorphism. By Lemma 2.2.2, it is enough to show that it is an isomorphism when applied to an object $\mathcal{M}$ of $\mathcal{C}(\mathcal{T}-\mathsf{Mod}_-)$ which is K-flat as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module. However, in this case $\mathscr{A}(\mathcal{M})$ is K-flat over $\mathcal{S}$ by Lemma 2.2.3, hence $L\Psi^* \circ \overline{\mathscr{A}}(\mathcal{M})$ is the image in the derived category of $\Psi^* \circ \mathscr{A}(\mathcal{M})$, which finishes the proof of (2.2.5).

Composing (2.2.4) with the restriction of (2.2.5) and then with the regrading functor ξ' we obtain an isomorphism of functors from $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_+)$ to $\mathcal{D}(\mathcal{R}'-\mathsf{Mod})^{\mathrm{op}}$

$$L\Psi^* \circ \kappa_\Omega \ \cong \ \kappa'_\Omega \circ R\Phi_*. \tag{2.2.7}$$

As the right hand side sends $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_+)$ to $\mathcal{D}^{\mathrm{bc}}(\mathcal{R}'-\mathsf{Mod}_-)^{\mathrm{op}}$, and as κ_Ω is an equivalence between $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}-\mathsf{Mod}_+)$ and $\mathcal{D}^{\mathrm{bc}}(\mathcal{R}-\mathsf{Mod}_-)^{\mathrm{op}}$, it follows from (2.2.7) that $L\Psi^*$ restricts to a functor from $\mathcal{D}^{\mathrm{bc}}(\mathcal{R}-\mathsf{Mod}_-)$ to $\mathcal{D}^{\mathrm{bc}}(\mathcal{R}'-\mathsf{Mod}_-)$. Then (2.2.7) proves the isomorphism of (1), hence finishes the proof. □

2.3. Application to intersection of subbundles. Now we will explain the geometric content of Proposition 2.1.1 in the context of §1.9. We let E and E' be vector bundles on X, and let

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & E' \arrow[dl] \\ & X & \end{tikzcd}$$

be a morphism of vector bundles over X. Let us stress that the morphism $X \to X$ induced by ϕ is assumed to be Id_X. We consider subbundles $F_1, F_2 \subseteq E$ and $F'_1, F'_2 \subseteq E'$, and assume that

$$\phi(F_1) \subseteq F'_1, \qquad \phi(F_2) \subseteq F'_2.$$

Let $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{E}'$, $\mathcal{F}'_1$, $\mathcal{F}'_2$ be the respective sheaves of sections of E, F_1, F_2, E', F'_1, F'_2. By Theorem 1.9.3 we have linear Koszul duality equivalences

$$\kappa_\Omega : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \ \xrightarrow{\sim}\ \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)^{\mathrm{op}},$$
$$\kappa'_\Omega : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F'_1 \overset{R}{\cap}_{E'} F'_2) \ \xrightarrow{\sim}\ \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F'_1)^\perp \overset{R}{\cap}_{(E')^*} (F'_2)^\perp)^{\mathrm{op}}.$$

We consider the complexes $\mathcal{X}$ (for the vector bundle E) and $\mathcal{X}'$ (for the vector bundle E') defined as in §1.9. The morphism ϕ defines a morphism of complexes $\mathcal{X}' \to \mathcal{X}$, to which we can apply (equivariant analogues of) the constructions of §2.1).

More geometrically, ϕ induces a morphism of dg-schemes $\Phi : F_1 \overset{R}{\cap}_E F_2 \to F'_1 \overset{R}{\cap}_{E'} F'_2$, and we have a (derived) direct image functor

$$R\Phi_* : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \to \mathcal{D}(\mathcal{T}'-\mathsf{Mod}).$$

This functor is just the restriction of the functor denoted similarly in §2.1 (in our special case).

Lemma 2.3.1. *Assume that the induced morphism of schemes between non-derived intersections $F_1 \cap_E F_2 \to F_1' \cap_{E'} F_2'$ is proper. Then the functor $R\Phi_*$ sends $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2)$ into the subcategory $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2')$.*

Proof. The dg-algebras $\mathcal{T}$ and $\mathcal{T}'$ are both concentrated in non-positive cohomological degrees; hence there exist natural morphisms of dg-algebras $\Theta : \mathcal{T} \to \mathcal{H}^0(\mathcal{T})$, $\Theta' : \mathcal{T}' \to \mathcal{H}^0(\mathcal{T}')$. Let also $\Phi^0 : \mathcal{H}^0(\mathcal{T}') \to \mathcal{H}^0(\mathcal{T})$ be the morphism induced by Φ, so that we have $\Phi^0 \circ \Theta' = \Theta \circ \Phi$. Taking direct images (i.e. restriction of scalars) we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{D}\big(\mathcal{H}^0(\mathcal{T})-\mathsf{Mod}\big) \arrow[r, "R\Phi^0_*"] \arrow[d, "\Theta_*"'] & \mathcal{D}\big(\mathcal{H}^0(\mathcal{T}')-\mathsf{Mod}\big) \arrow[d, "\Theta'_*"] \\
\mathcal{D}(\mathcal{T}-\mathsf{Mod}) \arrow[r, "R\Phi_*"] & \mathcal{D}(\mathcal{T}'-\mathsf{Mod}).
\end{tikzcd}$$

One can easily check that the functor Θ_* restricts to a functor from $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{H}^0(\mathcal{T})-\mathsf{Mod}\big)$ to $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod})$, and similarly for Θ'_*. Moreover, using Lemma 1.8.1 one can check that the essential image of Θ_* generates the category $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}-\mathsf{Mod})$. Hence it is enough to prove that $R\Phi^0_*$ sends $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{H}^0(\mathcal{T})-\mathsf{Mod}\big)$ into $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{H}^0(\mathcal{T}')-\mathsf{Mod}\big)$. However the morphism from $F_1 \cap_E F_2$ to X is affine, and the direct image of the structure sheaf under this morphism is $\mathcal{H}^0(\mathcal{T})$, so that we obtain an equivalence of categories

$$\mathcal{D}^{\mathrm{fg}}\big(\mathcal{H}^0(\mathcal{T})-\mathsf{Mod}\big) \ \cong \ \mathcal{D}^b\mathsf{Coh}^{\mathbb{G}_{\mathbf{m}}}(F_1 \cap_E F_2)$$

where $t \in \mathbb{G}_{\mathbf{m}}$ acts on E by dilatation by t^{-2} on the fibers, and on $F_1 \cap_E F_2$ by restriction. Similarly we have an equivalence

$$\mathcal{D}\big(\mathcal{H}^0(\mathcal{T}')-\mathsf{Mod}\big) \ \cong \ \mathcal{D}\mathsf{QCoh}^{\mathbb{G}_{\mathbf{m}}}(F_1' \cap_E F_2').$$

and under these equivalences the functor $R\Phi^0_*$ identifies with the (derived) direct image under the morphism $F_1 \cap_E F_2 \to F_1' \cap_{E'} F_2'$. Hence our claim follows from [H1, Proposition II.2.2]. □

We also consider the (derived) inverse image functor

$$L\Phi^* : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2') \to \mathcal{D}(\mathcal{T}-\mathsf{Mod}).$$

Again, this functor is the restriction of the functor denoted similarly in §2.1.

The morphism ϕ induces a morphism of vector bundles

$$\psi := \phi^\vee : (E')^* \to E^*,$$

which satisfies $\psi((F_i')^\perp) \subset F_i^\perp$ for $i = 1, 2$. Hence the above constructions and results also apply to ψ. We use similar notation.

The following result is an immediate application of Proposition 2.1.1 and Lemma 2.3.1.

Proposition 2.3.2. *Assume that X is a nice scheme admitting a dualizing complex Ω.*

(1) *Assume that the morphism of schemes* $F_1 \cap_E F_2 \to F_1' \cap_{E'} F_2'$ *induced by* ϕ *is proper. Then* $L\Psi^*$ *sends* $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)$ *into* $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F_1')^\perp \overset{R}{\cap}_{(E')^*} (F_2')^\perp)$*. Moreover, there exists a natural isomorphism of functors*

$$L\Psi^* \circ \kappa_\Omega \;\cong\; \kappa'_\Omega \circ R\Phi_* \quad : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \to \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F_1')^\perp \overset{R}{\cap}_{(E')^*} (F_2')^\perp)^{\mathrm{op}}.$$

(2) *Assume that the morphism of schemes* $(F_1')^\perp \cap_{(E')^*} (F_2')^\perp \to F_1^\perp \cap_{E'} F_2^\perp$ *induced by* ψ *is proper. Then* $L\Phi^*$ *sends* $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2')$ *into* $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2)$*. Moreover, there exists a natural isomorphism of functors*

$$\kappa_\Omega \circ L\Phi^* \;\cong\; R\Psi_* \circ \kappa'_\Omega \quad : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2') \to \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)^{\mathrm{op}}.$$

In particular, if both assumptions are satisfied, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \arrow[r, "\kappa_\Omega", "\sim"'] \arrow[d, shift right, "R\Phi_*"] & \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)^{\mathrm{op}} \arrow[d, shift left, "L\Psi^*"] \\
\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2') \arrow[r, "\kappa'_\Omega", "\sim"'] \arrow[u, shift right, "L\Phi^*"] & \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F_1')^\perp \overset{R}{\cap}_{(E')^*} (F_2')^\perp)^{\mathrm{op}}. \arrow[u, shift left, "R\Psi_*"]
\end{tikzcd}$$

3. Linear Koszul duality and base change

3.1. Statement. Let us come back to the setting of §1.7. More precisely, let X and Y be nice schemes, and let $\pi : X \to Y$ be a morphism of finite type. We assume that Y admits a dualizing complex Ω; then by [H1, Corollary VI.3.5], $\pi^!\Omega$ is a dualizing complex for X.

Let $\mathcal{X}$ be a complex of locally free sheaves on Y of the form considered in §1.2, and let $\mathcal{R}_Y, \mathcal{T}_Y, \mathcal{S}_Y$ be the associated dg-algebras. We also consider the complex of locally free sheaves $\pi^*\mathcal{X}$ on X, and let $\mathcal{R}_X, \mathcal{T}_X, \mathcal{S}_X$ be the associated dg-algebras. Note that we have natural isomorphism

$$\mathcal{R}_X \cong \pi^*\mathcal{R}_Y, \quad \mathcal{S}_X \cong \pi^*\mathcal{S}_Y, \quad \mathcal{T}_X \cong \pi^*\mathcal{T}_Y.$$

We denote by

$$\kappa^Y_\Omega : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}_Y\text{--}\mathsf{Mod}_+) \xrightarrow{\sim} \mathcal{D}^{\mathrm{bc}}(\mathcal{R}_Y\text{--}\mathsf{Mod}_-)^{\mathrm{op}}, \quad \kappa^X_{\pi^!\Omega} : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}_X\text{--}\mathsf{Mod}_+) \xrightarrow{\sim} \mathcal{D}^{\mathrm{bc}}(\mathcal{R}_X\text{--}\mathsf{Mod}_-)^{\mathrm{op}}$$

the associated equivalences of Theorem 1.7.1.

The morphism of schemes π induces a morphism of dg-ringed spaces $\hat{\pi} : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, and we can consider the associated direct and inverse image functors $\hat{\pi}_* : \mathcal{C}(\mathcal{T}_X\text{--}\mathsf{Mod}) \to \mathcal{C}(\mathcal{T}_Y\text{--}\mathsf{Mod})$, $\hat{\pi}^* : \mathcal{C}(\mathcal{T}_Y\text{--}\mathsf{Mod}) \to \mathcal{C}(\mathcal{T}_X\text{--}\mathsf{Mod})$. Note that the following diagram commutes:

(3.1.1)
$$\begin{tikzcd}
\mathcal{C}(\mathcal{T}_Y\text{--}\mathsf{Mod}) \arrow[r, "\hat{\pi}^*"] \arrow[d, "\mathrm{For}"'] & \mathcal{C}(\mathcal{T}_X\text{--}\mathsf{Mod}) \arrow[d, "\mathrm{For}"] \\
\mathcal{C}(\mathcal{O}_Y\text{--}\mathsf{Mod}) \arrow[r, "\pi^*"] & \mathcal{C}(\mathcal{O}_X\text{--}\mathsf{Mod}).
\end{tikzcd}$$

One can also consider the associated derived functors

$$R\hat{\pi}_* : \mathcal{D}(\mathcal{T}_X\text{--}\mathsf{Mod}) \to \mathcal{D}(\mathcal{T}_Y\text{--}\mathsf{Mod}), \qquad L\hat{\pi}^* : \mathcal{D}(\mathcal{T}_Y\text{--}\mathsf{Mod}) \to \mathcal{D}(\mathcal{T}_X\text{--}\mathsf{Mod})$$

(see §1.1).

In the following lemma we will say π has finite Tor-dimension if for any $\mathcal{F}$ in $\mathsf{QCoh}(Y)$ the set $\{i \in \mathbb{Z} \mid \mathcal{H}^i(L\pi^*\mathcal{F}) \neq 0\}$ is bounded. (Note that this terminology is not the one used in [H1]!)

Lemma 3.1.2. (1) *Assume that π has finite Tor-dimension. Then the functor $L\hat{\pi}^*$ restricts to a functor from $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}_Y{-}\mathsf{Mod}_{\pm})$ to $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}_X{-}\mathsf{Mod}_{\pm})$.*
(2) *Assume that π is proper. Then $R\hat{\pi}_*$ restricts to a functor from $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}_X{-}\mathsf{Mod}_+)$ to $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}_Y{-}\mathsf{Mod}_+)$.*

Proof. (1) The dg-algebra $\mathcal{T}_Y$ is K-flat as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module, hence any K-flat $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{T}_Y$-dg-module is also K-flat as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module (see [Ri, Lemma 1.3.2(ii)]). It follows (using (3.1.1)) that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(\mathcal{T}_Y{-}\mathsf{Mod}) \arrow[r, "L\hat{\pi}^*"] \arrow[d, "\mathrm{For}"'] & \mathcal{D}(\mathcal{T}_X{-}\mathsf{Mod}) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}(\mathcal{O}_Y{-}\mathsf{Mod}) \arrow[r, "L\pi^*"] & \mathcal{D}(\mathcal{O}_X{-}\mathsf{Mod}).
\end{tikzcd}$$

Hence we only have to prove the result when $\mathcal{X} = 0$, in which case it follows from [H1, Proposition II.4.4] and our assumption on π.

The proof of (2) is similar, using the compatibility of the functor $R\hat{\pi}_*$ with $R\pi_*$ (see [BR, Proposition 3.3.6]) and [H1, Proposition II.2.2]. □

In addition to the above functors, we will also consider the following one, in case π has finite Tor dimension:

$$\hat{\pi}^! := \mathbf{D}^{\mathcal{T}_X}_{\pi^!\Omega} \circ L\hat{\pi}^* \circ \mathbf{D}^{\mathcal{T}_Y}_{\Omega} : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}_Y{-}\mathsf{Mod}_+) \to \mathcal{D}^{\mathrm{bc}}(\mathcal{T}_X{-}\mathsf{Mod}_+)$$

(see §1.5 for the definition of duality functors).

Similarly, π induces a morphism of dg-ringed spaces $\tilde{\pi} : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$. By the same arguments as above, if π has finite Tor-dimension then we have a derived inverse image functor

$$L\tilde{\pi}^* : \mathcal{D}^{\mathrm{bc}}(\mathcal{R}_Y{-}\mathsf{Mod}_-) \to \mathcal{D}^{\mathrm{bc}}(\mathcal{R}_X{-}\mathsf{Mod}_-)$$

and, if π is proper, we have a derived direct image functor

$$R\tilde{\pi}_* : \mathcal{D}^{\mathrm{bc}}(\mathcal{R}_X{-}\mathsf{Mod}_-) \to \mathcal{D}^{\mathrm{bc}}(\mathcal{R}_Y{-}\mathsf{Mod}_-).$$

The following result expresses the compatibility of our Koszul duality equivalence with base change. It is similar in spirit to [Ri, Proposition 2.4.5].

Proposition 3.1.3. *Assume that X and Y are nice schemes, that π is of finite type, and that Y admits a dualizing complex Ω.*

(1) *Assume moreover that π has finite Tor-dimension. Then there exists an isomorphism of functors:*
$$L\tilde{\pi}^* \circ \kappa^Y_{\Omega} \;\cong\; \kappa^X_{\pi^!\Omega} \circ \hat{\pi}^! \quad : \mathcal{D}^{\mathrm{bc}}(\mathcal{T}_Y{-}\mathsf{Mod}_+) \to \mathcal{D}^{\mathrm{bc}}(\mathcal{R}_X{-}\mathsf{Mod}_-)^{\mathrm{op}}.$$
(2) *Assume moreover that π is proper. Then there exists an isomorphism of functors:*
$$R\tilde{\pi}_* \circ \kappa^X_{\pi^!\Omega} \;\cong\; \kappa^Y_{\Omega} \circ R\hat{\pi}_* \quad : \mathcal{D}^{\mathrm{bc}}(\mathcal{R}_X{-}\mathsf{Mod}_-) \to \mathcal{D}^{\mathrm{bc}}(\mathcal{T}_Y{-}\mathsf{Mod}_+)^{\mathrm{op}}.$$

3.2. Proof of Proposition 3.1.3. We begin by proving some compatibility results for the equivalences $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$ of §1.4 (under weaker assumptions than in the proposition). We use subscripts to indicate the scheme we are working on. The morphism π induces a morphism of dg-ringed spaces $\check{\pi} : (X, \mathcal{S}_X) \to (Y, \mathcal{S}_Y)$. As above, we have corresponding derived direct and inverse image functors[5]

$$R\check{\pi}_* : \mathcal{D}(\mathcal{S}_X\text{-}\mathsf{Mod}) \to \mathcal{D}(\mathcal{S}_Y\text{-}\mathsf{Mod}), \qquad L\check{\pi}^* : \mathcal{D}(\mathcal{S}_Y\text{-}\mathsf{Mod}) \to \mathcal{D}(\mathcal{S}_X\text{-}\mathsf{Mod}).$$

Using the same arguments as in the proof of Lemma 3.1.2, one can check that these functors restrict to functors

$$R\check{\pi}_* : \mathcal{D}(\mathcal{S}_X\text{-}\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{S}_Y\text{-}\mathsf{Mod}_-), \qquad L\check{\pi}^* : \mathcal{D}(\mathcal{S}_Y\text{-}\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{S}_X\text{-}\mathsf{Mod}_-)$$

which form an adjoint pair, and that the functors $R\hat{\pi}_*$ and $L\hat{\pi}^*$ restrict to functors

$$R\hat{\pi}_* : \mathcal{D}(\mathcal{T}_X\text{-}\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{T}_Y\text{-}\mathsf{Mod}_-), \qquad L\hat{\pi}^* : \mathcal{D}(\mathcal{T}_Y\text{-}\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{T}_X\text{-}\mathsf{Mod}_-)$$

which also form an adjoint pair.

Proposition 3.2.1. *Assume that X and Y are nice schemes.*

(1) *There exists an isomorphism of functors:*

$$L\check{\pi}^* \circ \overline{\mathscr{A}}_Y \;\cong\; \overline{\mathscr{A}}_X \circ L\hat{\pi}^* \quad : \mathcal{D}(\mathcal{T}_Y\text{-}\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{S}_X\text{-}\mathsf{Mod}_-).$$

(2) *There exists an isomorphism of functors:*

$$\overline{\mathscr{A}}_Y \circ R\hat{\pi}_* \;\cong\; R\check{\pi}_* \circ \overline{\mathscr{A}}_X \quad : \mathcal{D}(\mathcal{T}_Y\text{-}\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{S}_X\text{-}\mathsf{Mod}_-).$$

Proof. (1) By Theorem 1.4.1, it is equivalent to prove an isomorphism of functors:

$$\overline{\mathscr{B}}_X \circ L\check{\pi}^* \;\cong\; L\hat{\pi}^* \circ \overline{\mathscr{B}}_Y \quad : \mathcal{D}(\mathcal{S}_Y\text{-}\mathsf{Mod}_-) \to \mathcal{D}(\mathcal{T}_X\text{-}\mathsf{Mod}_-).$$

Using the same arguments as for (2.2.6) one can easily construct a morphism of functors from the right hand side to the left side; what remains is to prove that it is an isomorphism.

By construction of resolutions, there are enough objects in $\mathcal{C}(\mathcal{S}_Y\text{-}\mathsf{Mod}_-)$ which are K-flat as $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{S}_Y$-dg-modules. Hence it is enough to prove that for any object $\mathcal{N}$ of $\mathcal{C}(\mathcal{S}_Y\text{-}\mathsf{Mod}_-)$ which is K-flat as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{S}_Y$-dg-module, with image $\overline{\mathcal{N}}$ in the derived category, our morphism

$$L\hat{\pi}^* \circ \overline{\mathscr{B}}_Y(\overline{\mathcal{N}}) \;\to\; \overline{\mathscr{B}}_X \circ L\check{\pi}^*(\overline{\mathcal{N}}) \tag{3.2.2}$$

is an isomorphism.

The right hand side of (3.2.2) is easy to compute: it is isomorphic to the image in the derived category of the $\mathcal{T}_X$-dg-module $\mathscr{B}_X\big(\check{\pi}^*\mathcal{N}\big)$. Let us consider now the right hand side. As $\mathcal{S}_Y$ is K-flat as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_Y$-dg-module (see Lemma 2.2.1), any K-flat $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{S}_Y$-dg-module is also K-flat as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_Y$-dg-module (see [Ri, Lemma 1.3.2]); in particular $\mathcal{N}$ has this property. As checked in the course of the proof of Lemma 2.2.2, this implies that $\mathscr{B}_Y(\mathcal{N})$ is K-flat as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_Y$-dg-module. From this it follows that the left hand side of (3.2.2) is isomorphic to the image in the derived category of $\hat{\pi}^*\big(\mathscr{B}_Y(\mathcal{N})\big)$. Indeed, let $\mathcal{P}$ be a K-flat resolution of $\mathscr{B}_Y(\mathcal{N})$ as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{T}_Y$-dg-module; then the

[5]Note that $\mathcal{S}_X$ and $\mathcal{S}_Y$ are not non-positively graded, so that we cannot apply directly the results quoted in §1.1. However, one can reduce our situation to the one of §1.1 using the regrading equivalence of §1.7.

morphism $\hat{\pi}^*\mathcal{P} \to \hat{\pi}^*\big(\mathscr{B}_Y(\mathcal{N})\big)$ is a quasi-isomorphism since the functor $\hat{\pi}^*$ sends every acyclic $\mathcal{T}_Y$-dg-module which is K-flat as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_Y$-dg-module to an acyclic dg-module.

Using these remarks, the fact that (3.2.2) is an isomorphism now follows from the natural isomorphism $\mathscr{B}_X\big(\hat{\pi}^*\mathcal{N}\big) \cong \hat{\pi}^*\big(\mathscr{B}_Y(\mathcal{N})\big)$, which finishes the proof of (1).

Isomorphism (2) follows from (1) by adjunction. □

We will also need the following compatibility result for direct images and duality. It is a "dg-version" of Hartshorne's Duality Theorem of [H1].

Proposition 3.2.3. *Assume that X and Y are nice schemes, that π is of finite type, and that Y admits a dualizing complex Ω. Assume also that π is proper. Then there exists an isomorphism of functors from $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}_X-\mathsf{Mod})$ to $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}_Y-\mathsf{Mod})^{\mathrm{op}}$:*

$$R\hat{\pi}_* \circ \mathbf{D}^{\mathcal{T}_X}_{\pi^!\Omega} \ \cong\ \mathbf{D}^{\mathcal{T}_Y}_{\Omega} \circ R\hat{\pi}_*.$$

Proof. In this proof we will work with some dg-modules which are not necessarily quasi-coherent, but which have quasi-coherent cohomology. This is allowed by [BR, Proposition 3.3.2]. Note also that by [BR, Proposition 3.3.6] we can compute the image under $R\hat{\pi}_*$ of an object of $\mathcal{D}(\mathcal{T}_X-\mathsf{Mod})$ in the larger derived category of all sheaves of $\mathcal{T}_X$-dg-modules. Finally, in this proof we write $Q\mathcal{M}$ for the image in the suitable derived category of a dg-module $\mathcal{M}$.

Recall that we have fixed bounded below complexes of injective quasi-coherent $\mathcal{O}_X$-modules $\mathcal{I}_\Omega$ and $\mathcal{I}_{\pi^!\Omega}$ such that $Q\mathcal{I}_\Omega \cong \Omega$ and $Q\mathcal{I}_{\pi^!\Omega} \cong \pi^!\Omega$. By [H1, p. 257], we can assume that both of them are bounded complexes. Note that, as X and Y are in particular locally Noetherian, the terms of these complexes are injective as $\mathcal{O}$-modules (see [H1, Proposition II.7.17]).

Start with some $\mathcal{M}$ in $\mathcal{C}(\mathcal{T}_X-\mathsf{Mod})$ such that $Q\mathcal{M}$ is in $\mathcal{D}^{\mathrm{bc}}(\mathcal{T}_X-\mathsf{Mod})$, and denote by $\mathcal{J}$ a K-injective object of $\mathcal{C}(\mathcal{T}_X-\mathsf{Mod})$ endowed with a quasi-isomorphism $\mathcal{M} \xrightarrow{\mathrm{qis}} \mathcal{J}$. Then we have

$$R\hat{\pi}_* \circ \mathbf{D}^{\mathcal{T}_X}_{\pi^!\Omega}(Q\mathcal{M}) \ \cong\ Q\hat{\pi}_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J},\mathcal{I}_{\pi^!\Omega}). \tag{3.2.4}$$

Indeed, by definition we have $\mathbf{D}^{\mathcal{T}_X}_{\pi^!\Omega}(Q\mathcal{M}) \cong Q\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{I}_{\pi^!\Omega})$. Now as $\mathcal{I}_{\pi^!\Omega}$ is a bounded complex of injective $\mathcal{O}_X$-modules, it is K-injective in the category of all $\mathcal{O}_X$-modules, hence the morphism $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J},\mathcal{I}_{\pi^!\Omega}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{I}_{\pi^!\Omega})$ is a quasi-isomorphism (see [Li, Lemma (2.4.5.1)]). Now again as $\mathcal{I}_{\pi^!\Omega}$ is a bounded complex of injective $\mathcal{O}_X$-module, the complex of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J},\mathcal{I}_{\pi^!\Omega})$ is made of flasque sheaves, which are π_*-acyclic. By [Li, Corollary (3.9.3.5)], we deduce that the natural morphism $Q\pi_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J},\mathcal{I}_{\pi^!\Omega}) \to R\pi_*Q\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J},\mathcal{I}_{\pi^!\Omega})$ is an isomorphism. One can deduce (3.2.4) from this.

Now there exists a morphism of dg-modules

$$\hat{\pi}_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J},\mathcal{I}_{\pi^!\Omega}) \to \mathcal{H}om_{\mathcal{O}_Y}(\hat{\pi}_*\mathcal{J},\pi_*\mathcal{I}_{\pi^!\Omega}).$$

Composing with the morphism induced by the "trace morphism" $\pi_*\mathcal{I}_{\pi^!\Omega} \to \mathcal{I}_\Omega$ one obtains a morphism of dg-modules

$$\hat{\pi}_*\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{J},\mathcal{I}_{\pi^!\Omega}) \to \mathcal{H}om_{\mathcal{O}_Y}(\hat{\pi}_*\mathcal{J},\mathcal{I}_\Omega).$$

Moreover, the image in the derived category of the right hand side is isomorphic to $\mathbf{D}^{\mathcal{T}_Y}_{\Omega} \circ R\hat{\pi}_*(Q\mathcal{M})$. Hence we have constructed a natural morphism

$$R\hat{\pi}_* \circ \mathbf{D}^{\mathcal{T}_X}_{\pi^!\Omega}(Q\mathcal{M}) \ \to \ \mathbf{D}^{\mathcal{T}_Y}_{\Omega} \circ R\hat{\pi}_*(Q\mathcal{M}).$$

After forgetting the action of $\mathcal{T}_Y$, this is the "duality morphism" of [H1, p. 378], which is known to be an isomorphism (see [H1, Theorem VII.3.3]). Hence this morphism is also an isomorphism, which finishes the proof of the proposition. □

Proof of Proposition 3.1.3. (1) We obtain the isomorphism as the following composition of isomorphisms of functors:

$$\begin{array}{rcl} L\tilde{\pi}^* \circ \kappa^Y_{\Omega} & = & L\tilde{\pi}^* \circ \xi_Y \circ \overline{\mathscr{A}}^{\mathrm{bc}}_Y \circ \mathbf{D}^{\mathcal{T}_Y}_{\Omega} \\ & \cong & \xi_X \circ L\check{\pi}^* \circ \overline{\mathscr{A}}^{\mathrm{bc}}_Y \circ \mathbf{D}^{\mathcal{T}_Y}_{\Omega} \\ & \overset{\text{Prop. 3.2.1}}{\cong} & \xi_X \circ \overline{\mathscr{A}}^{\mathrm{bc}}_X \circ L\hat{\pi}^* \circ \mathbf{D}^{\mathcal{T}_Y}_{\Omega} \\ & = & \kappa^X_{\pi^!\Omega} \circ \hat{\pi}^!. \end{array}$$

(2) We obtain the isomorphism as the following composition of isomorphisms of functors:

$$\begin{array}{rcl} R\tilde{\pi}_* \circ \kappa^X_{\pi^!\Omega} & = & R\tilde{\pi}_* \circ \xi_X \circ \overline{\mathscr{A}}^{\mathrm{bc}}_X \circ \mathbf{D}^{\mathcal{T}_X}_{\pi^!\Omega} \\ & \cong & \xi_Y \circ R\check{\pi}_* \circ \overline{\mathscr{A}}^{\mathrm{bc}}_X \circ \mathbf{D}^{\mathcal{T}_X}_{\pi^!\Omega} \\ & \overset{\text{Prop. 3.2.1}}{\cong} & \xi_Y \circ \overline{\mathscr{A}}^{\mathrm{bc}}_Y \circ R\hat{\pi}_* \circ \mathbf{D}^{\mathcal{T}_X}_{\pi^!\Omega} \\ & \overset{\text{Prop. 3.2.3}}{\cong} & \kappa^Y_{\Omega} \circ R\hat{\pi}_*. \end{array}$$

□

3.3. Compatibility with inverse image in the case of smooth varieties. The isomorphism of Proposition 3.1.3(1) is not easy to work with in general since the functor $\hat{\pi}^!$ has not a completely explicit description. In this subsection we give an easier description of this functor in the case X and Y are Noetherian, integral, separated, regular schemes of finite dimension. In this case π has automatically finite Tor-dimension, as follows from [H2, Ex. III.6.9]. Moreover Ω is a shift of a line bundle (see [H1, Theorem V.3.1 and §V.10]), hence $\pi^*\Omega$ is also a dualizing complex on X. For simplicity we will also assume that $n \leq 1$ and work with locally finitely generated dg-modules, so that we are in the setting of §1.8.

Lemma 3.3.1. *For any $\mathcal{M}$ in $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}_Y-\mathsf{Mod})$, there exists an object $\mathcal{P}$ in $\mathcal{C}(\mathcal{T}_Y-\mathsf{Mod})$ such that $\mathcal{P}_j$ is a finite complex of locally free $\mathcal{O}_Y$-modules of finite rank for any $j \in \mathbb{Z}$, and whose image in $\mathcal{D}(\mathcal{T}_Y-\mathsf{Mod})$ is isomorphic to $\mathcal{M}$.*

Proof. By Lemma 1.8.1 we can assume that $\mathcal{M}$ is in $\mathcal{CFG}(\mathcal{T}_Y)$. Then the construction of the proof of [MR1, Proposition 3.1.1] produces an object $\mathcal{P}$ as in the statement and a quasi-isomorphism $\mathcal{P} \xrightarrow{\mathrm{qis}} \mathcal{M}$. □

Proposition 3.3.2. *Assume that X and Y are Noetherian, integral, separated, regular schemes of finite dimension. Then there exists an isomorphism of functors*

$$L\hat{\pi}^* \circ \mathbf{D}^{\mathcal{T}_Y}_{\mathcal{E}} \ \cong \ \mathbf{D}^{\mathcal{T}_X}_{\pi^*\mathcal{E}} \circ L\hat{\pi}^* \quad : \mathcal{D}^{\mathrm{fg}}(\mathcal{T}_Y-\mathsf{Mod}) \to \mathcal{D}^{\mathrm{bc}}(\mathcal{T}_X-\mathsf{Mod})^{\mathrm{op}}.$$

Proof. Recall that by [BR, §3.2–3.3] one can compute the derived functors of inverse and direct image functors also in the categories $\widetilde{\mathcal{D}}$ of all sheaves of dg-modules, and that we obtain functors which are compatible in the natural sense with their version for categories $\mathcal{D}$.

First we construct a morphism of functors

$$\mathbf{D}_{\Omega}^{\mathcal{T}_Y} \ \to \ R\hat{\pi}_* \circ \mathbf{D}_{\pi^*\Omega}^{\mathcal{T}_X} \circ L\hat{\pi}^* \quad : \widetilde{\mathcal{C}}(\mathcal{T}_Y-\mathsf{Mod}) \to \widetilde{\mathcal{C}}(\mathcal{T}_Y-\mathsf{Mod})^{\mathrm{op}}. \tag{3.3.3}$$

For this construction we observe that we have an isomorphism of functors

$$\pi_* \mathcal{H}om_{\mathcal{O}_X}(\pi^*(-), \pi^*\Omega) \ \cong \ \mathcal{H}om_{\mathcal{O}_Y}(-, \pi_*\pi^*\Omega)$$

by the local version of the $(-)^*$-$(-)_*$ adjunction (see e.g. [KS, Corollary 2.3.4]), so that we obtain a morphism of functors

$$\mathcal{H}om_{\mathcal{O}_Y}(-, \Omega) \ \to \ \hat{\pi}_* \mathcal{H}om_{\mathcal{O}_X}(\hat{\pi}^*(-), \pi^*\Omega) \quad : \widetilde{\mathcal{C}}(\mathcal{T}_Y-\mathsf{Mod}) \to \widetilde{\mathcal{C}}(\mathcal{T}_Y-\mathsf{Mod})^{\mathrm{op}}$$

Now we observe that the right derived functor of the left hand side is the left hand side of (3.3.3), while the right hand side is the composition of the functors $\hat{\pi}^* : \widetilde{\mathcal{C}}(\mathcal{T}_Y-\mathsf{Mod})^{\mathrm{op}} \to \widetilde{\mathcal{C}}(\mathcal{T}_X-\mathsf{Mod})^{\mathrm{op}}$, $\mathcal{H}om_{\mathcal{O}_X}(-, \pi^*\Omega) : \widetilde{\mathcal{C}}(\mathcal{T}_X-\mathsf{Mod})^{\mathrm{op}} \to \widetilde{\mathcal{C}}(\mathcal{T}_X-\mathsf{Mod})$ and $\hat{\pi}_* : \widetilde{\mathcal{C}}(\mathcal{T}_X-\mathsf{Mod}) \to \widetilde{\mathcal{C}}(\mathcal{T}_Y-\mathsf{Mod})$. These functors admit right derived functors, and the composition of these derived functors are the right hand side of (3.3.3). Hence we obtain morphism (3.3.3) by standard properties of (right) derived functors.

By adjunction, from (3.3.3) we obtain a morphism of functors

$$L\hat{\pi}^* \circ \mathbf{D}_{\Omega}^{\mathcal{T}_Y} \ \to \ \mathbf{D}_{\pi^*\Omega}^{\mathcal{T}_X} \circ L\hat{\pi}^* \quad : \widetilde{\mathcal{C}}(\mathcal{T}_Y-\mathsf{Mod}) \to \widetilde{\mathcal{C}}(\mathcal{T}_X-\mathsf{Mod})^{\mathrm{op}}. \tag{3.3.4}$$

To conclude the proof, we only have to check that this morphism is an isomorphism on objects of $\mathcal{D}^{\mathrm{fg}}(\mathcal{T}_Y-\mathsf{Mod})$. By Lemma 3.3.1 it is enough to prove that it is an isomorphism on images in the derived category of objects $\mathcal{P}$ in $\mathcal{C}(\mathcal{T}_Y-\mathsf{Mod})$ such that $\mathcal{P}_j$ is a finite complex of locally free $\mathcal{O}_Y$-modules of finite rank for any $j \in \mathbb{Z}$. Let $\mathcal{P}$ be such an object. It is easy to check that $L\hat{\pi}^*\mathcal{P}$ is the image in the derived category of $\hat{\pi}^*\mathcal{P}$, that the natural morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\hat{\pi}^*\mathcal{P}, \pi^*\Omega) \ \to \ \mathcal{H}om_{\mathcal{O}_X}(\hat{\pi}^*\mathcal{P}, \mathcal{I}_{\pi^*\Omega})$$

is a quasi-isomorphism, and finally that we have an isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\hat{\pi}^*\mathcal{P}, \pi^*\Omega) \ \cong \ \hat{\pi}^*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \Omega).$$

Hence $\mathbf{D}_{\pi^*\Omega}^{\mathcal{T}_X} \circ L\hat{\pi}^*\mathcal{P}$ is the image in the derived category of $\hat{\pi}^*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \Omega)$. By similar arguments, one can check that $L\hat{\pi}^* \circ \mathbf{D}_{\Omega}^{\mathcal{T}_Y}$ is also the image in the derived category of $\hat{\pi}^*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \Omega)$, and that (3.3.4) applied to $\mathcal{P}$ is an isomorphism. □

Combining Proposition 3.2.1 and Proposition 3.3.2 we obtain the following result.

Corollary 3.3.5. *Assume that X and Y are Noetherian, integral, separated, regular schemes of finite dimension. Then there exists an isomorphism of functors*

$$L\tilde{\pi}^* \circ \kappa_{\Omega}^{Y} \ \cong \ \kappa_{\pi^*\Omega}^{X} \circ L\hat{\pi}^* \quad : \mathcal{D}^{\mathrm{fg}}(\mathcal{T}_Y-\mathsf{Mod}) \to \mathcal{D}^{\mathrm{fg}}(\mathcal{R}_X-\mathsf{Mod})^{\mathrm{op}}.$$

3.4. Application to intersections of subbundles. Now we explain the geometric content on Propositions 3.1.3 and 3.3.2 in the context of §1.9. We assume as above that X and Y are nice schemes, that $\pi : X \to Y$ is a morphism of finite type, and that Y admits a dualizing complex Ω.

Consider a vector bundle E on Y, and let $F_1, F_2 \subset E$ be subbundles. Consider also $E^X := E \times_Y X$, which is a vector bundle on X, and the subbundles $F_i^X := F_i \times_Y X \subset E^X$ $(i = 1, 2)$. If $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$ are the respective sheaves of sections of E, F_1, F_2, then $\pi^*\mathcal{E}$, $\pi^*\mathcal{F}_1$, $\pi^*\mathcal{F}_2$ are the sheaves of sections of E^X, F_1^X, F_2^X, respectively. Out of these data we define the complexes $\mathcal{X}_X$ and $\mathcal{X}_Y$ as in §1.9, and then the dg-algebras $\mathcal{T}_X$, $\mathcal{S}_X$, $\mathcal{R}_X$ and $\mathcal{T}_Y$, $\mathcal{S}_Y$, $\mathcal{R}_Y$. Note that we have natural isomorphisms of dg-algebras

$$\mathcal{T}_X \cong \pi^*\mathcal{T}_Y, \quad \mathcal{S}_X \cong \pi^*\mathcal{S}_Y, \quad \mathcal{R}_X \cong \pi^*\mathcal{R}_Y.$$

We define the categories

$$\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2), \quad \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)$$
$$\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X), \quad \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp)$$

as in §1.9. Then by Theorem 1.9.3 there are equivalences of categories

$$\kappa^X_{\pi^!\Omega} : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp)^{\mathrm{op}},$$
$$\kappa^Y_{\Omega} : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)^{\mathrm{op}}.$$

If X and Y are Noetherian, integral, separated, regular schemes of finite dimension, we also have an equivalence

$$\kappa^X_{\pi^*\Omega} : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp)^{\mathrm{op}}.$$

The morphism of schemes π induces a morphism of dg-schemes

$$\hat{\pi} : F_1^X \overset{R}{\cap}_{E^X} F_2^X \to F_1 \overset{R}{\cap}_E F_2.$$

This morphism can be represented by the natural morphism of dg-ringed spaces $(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$.

Lemma 3.4.1. (1) *Assume π has finite Tor-dimension. The functor*

$$L\hat{\pi}^* : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \to \mathcal{D}(\mathcal{T}_X\text{−}\mathsf{Mod})$$

takes values in $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X)$.

(2) *Assume π is proper. Then the functor*

$$R\hat{\pi}_* : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X) \to \mathcal{D}(\mathcal{T}_Y\text{−}\mathsf{Mod})$$

takes values in $\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2)$.

Proof. (1) As $\mathcal{T}_X \cong \pi^*\mathcal{T}_Y$ and $\mathcal{T}_Y$ is K-flat over $\mathcal{T}_Y^0 \cong \mathrm{S}(\mathcal{F}_2^\vee)$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \arrow[r, "L\hat{\pi}^*"] \arrow[d, "\mathrm{For}"] & \mathcal{D}(\mathcal{T}_X\text{−}\mathsf{Mod}) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^{\mathrm{fg}}(\mathrm{S}(\mathcal{F}_2^\vee)\text{−}\mathsf{Mod}) \arrow[r, "L\overline{\pi}^*"] & \mathcal{D}(\mathrm{S}(\pi^*\mathcal{F}_2^\vee)\text{−}\mathsf{Mod}).
\end{tikzcd}$$

On the bottom line, $\overline{\pi}$ is the morphism of dg-schemes $(X,\, \mathrm{S}(\pi^*\mathcal{F}_2^\vee)) \to (Y,\, \mathrm{S}(\mathcal{F}_2^\vee))$ induced by π. But $\mathcal{D}^{\mathrm{fg}}(\mathrm{S}(\mathcal{F}_2^\vee)\text{−}\mathsf{Mod})$ is naturally equivalent to $\mathcal{D}^b\mathsf{Coh}^{\mathbb{G}_{\mathbf{m}}}(F_2)$ (see the arguments in the proof of Lemma 2.3.1), and $\mathcal{D}(\mathrm{S}(\pi^*\mathcal{F}_2^\vee)\text{−}\mathsf{Mod})$ to $\mathcal{D}\mathsf{QCoh}^{\mathbb{G}_{\mathbf{m}}}(F_2^X)$. Moreover, via these indentifications, $L\overline{\pi}^*$ is the inverse image functor for the morphism $F_2^X \to F_2$ induced by π. Hence $L\overline{\pi}^*$ takes values in $\mathcal{D}^{\mathrm{fg}}(\mathrm{S}(\pi^*\mathcal{F}_2^\vee)\text{−}\mathsf{Mod})$. Our result follows.

(2) The proof is similar. It uses the fact that, as $\mathcal{T}_X$ is K-flat over $\mathrm{S}(\pi^*\mathcal{F}_2^\vee)$, every K-injective object in $\mathcal{C}(\mathcal{T}_X\text{−}\mathsf{Mod})$ has an image in $\mathcal{C}(\mathrm{S}(\mathcal{F}_2^\vee)\text{−}\mathsf{Mod})$ which is also K-injective (see [Ri, Lemma 1.3.2]), so that the diagram

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X) \arrow[r, "R\hat{\pi}^*"] \arrow[d, "\mathrm{For}"] & \mathcal{D}(\mathcal{T}_Y\text{−}\mathsf{Mod}) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^{\mathrm{fg}}(\mathrm{S}(\pi^*\mathcal{F}_2^\vee)\text{−}\mathsf{Mod}) \arrow[r, "R\overline{\pi}^*"] & \mathcal{D}(\mathrm{S}(\mathcal{F}_2^\vee)\text{−}\mathsf{Mod})
\end{tikzcd}$$

commutes. □

Similarly, π induces a morphism of dg-schemes

$$\tilde{\pi} : (F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp \to F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp,$$

hence, if π has finite Tor-dimension, a functor

$$L\tilde{\pi}^* : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp) \ \to\ \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp),$$

and, if π is proper, a functor

$$R\tilde{\pi}_* : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp) \ \to\ \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp).$$

The following result is an immediate consequence of Proposition 3.1.3 and Corollary 3.3.5.

Proposition 3.4.2. *Assume that X and Y are nice schemes, that π is of finite type, and that Y admits a dualizing complex Ω.*

(1) *If π is proper, there exists a natural isomorphism of functors*

$$\kappa_\Omega^Y \circ R\hat{\pi}_* \ \cong\ R\tilde{\pi}_* \circ \kappa^X_{\pi^!\Omega} \quad : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X) \to \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)^{\mathrm{op}}.$$

(2) *If X and Y are Noetherian, integral, separated, regular schemes of finite dimension, then there exists a natural isomorphism of functors*

$$L\tilde{\pi}^* \circ \kappa_\Omega^Y \ \cong\ \kappa^X_{\pi^*\Omega} \circ L\hat{\pi}^* \quad : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \to \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp)^{\mathrm{op}}.$$

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